THE UNIFORMIZATION THEOREM

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Abstract. We discuss a proof of the uniformization theorem.

1. Introduction

In these notes we will prove the uniformization theorem:

Theorem 1.1 (uniformization theorem). *Let X be a simply connected Riemann surface. Then up to isomorphism, either $X = \mathbf{C}$, $X = \mathbf{D}$, or $X = \mathbf{P}^1$.*

The significance of the uniformization theorem is that it totally classifies universal covers of Riemann surfaces.

There are a handful of proofs of the uniformization theorem. Many involve identifying the complex structure on X with a conformal class of Riemannian metrics on X, and then classifying the Riemannian metrics on X, either by an exhaustion argument or by running the Ricci flow on X. However, the proof that we give is the one that appears in Forster [GF12, Chapter 3], and is almost purely complex-analytic, though it does appeal to some functional analysis.

It will be convenient to prove the uniformization theorem with a slightly different hypothesis. If X is compact, then X has genus 0, and hence is isomorphic to $\mathbf{P}^1$, so we might as well assume that X is noncompact.

Let d' be the *holomorphic* differential, so $d'f$ is locally $f(z)\ dz$ for a holomorphic function f, and $d'\omega = 0$ for a holomorphic 1-form ω.

Definition 1.2. The *holomorphic de Rham cohomology* $H^\bullet_{\mathscr{O}}(X, \mathbf{C})$ of X is the cohomology of the cochain complex defined by the boundary map d'.

If X is simply connected and ω is a holomorphic 1-form, then we may fix $z_0 \in X$ and define f by

$$f(z) = \int_{z_0}^{z} \omega$$

where the choice of path does not matter because every curve in X is contractible. Therefore $H^1_{\mathscr{O}}(X, \mathbf{C}) = 0$. Thus it suffices to show:

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Proposition 1.3. *Let X be a noncompact Riemann surface with $H^1_{\mathscr{O}}(X, \mathbf{C}) = 0$. Then up to isomorphism, either $X = \mathbf{C}$, $X = \mathbf{D}$, or $X = \mathbf{P}^1$.*

2. Preliminaries

2.1. **Functional analysis.** By a test function on Y we mean a smooth function with compact support in Y. We write $\mathscr{D}$, $\mathscr{E}$, $\mathscr{O}$, and $\mathscr{M}$ for the presheaves of compactly supported smooth, smooth, holomorphic, and meromorphic functions respectively. Let Ω be the sheaf of holomorphic 1-forms. All of these except $\mathscr{D}$ are actually sheaves.

On any Riemann surface, the Laplace equation can be written as $d'd''f = 0$; in coordinates it can be written $\Delta f = 0$. Solutions of the Laplace equation are called harmonic.

Theorem 2.1. *Let Y be an open subset of $\mathbf{C}$. Suppose that for every $y \in \partial Y$ there is a disk D which does not meet $\overline{Y}$, such that $y \in \partial D$. Then the Dirichlet problem for the Laplace equation on Y is well-posed.*

For the proof, see Forster [GF12, Theorem 22.18].

We turn $\mathscr{D}$ into a presheaf of topological vector spaces by declaring that in $\mathscr{D}(Y)$, a sequence f_n converges to f if there is a compact $K \subseteq Y$ such that $\operatorname{supp} f \cup \bigcup_n \operatorname{supp} f_n \subseteq K$ and for every linear differential operator P on K with constant coefficients, $Pf_n \to Pf$. We let $\mathscr{D}'$ be the dual sheaf of $\mathscr{D}$; that is, $\mathscr{D}'(Y)$ is the topological vector space of bounded linear maps $\mathscr{D}(Y) \to \mathbf{C}$. We call $\mathscr{D}'$ the sheaf of distributions.

Definition 2.2. A *holomorphic distribution* f is one such that for every $g \in \mathscr{D}(Y)$, $\langle f, \overline{\partial} g\rangle = 0$.

Theorem 2.3 (Weyl's elliptic regularity lemma)**.** *Let f be a holomorphic distribution on Y. Then there is a holomorphic function, which we also denote f, such that for every $g \in \mathscr{D}(Y)$,*

$$\int_Y fg \; dV = \langle f, g\rangle.$$

For the proof, see Evans [Eva10, §2.2], who proves it for *harmonic* distributions.

We recall that since the Cauchy-Riemann operator d'' is elliptic, we can locally invert it in the following sense. For every $\omega \in \mathscr{E}^{0,1}(X)$ and $Y \Subset X$, we can find $f \in \mathscr{E}(Y)$ such that $d''f = \omega|Y$. For the details, see Forster [GF12, Corollary 14.16]. Alternatively, one can use Hadamard's parametrix construction [Hör94, Theorem 17.1.1'].

We turn $\mathscr{E}$ into a sheaf of Fréchet spaces, under the seminorms $u \mapsto ||\partial^\alpha u||_{L^\infty(K)}$ whenever K is a compact set contained in a coordinate chart. Then every linear map $\mathscr{E}(Y) \to \mathbf{C}$ has compact support. So the dual presheaf $\mathscr{E}'$ of topological vector spaces is called the sheaf of compactly supported distributions. Similarly we define $(\mathscr{E}')^{0,1}$.

We turn $\mathscr{O}$ into a sheaf of Fréchet spaces, by restricting the topology from L^∞_{loc}.

We will need the following form of the Hanh-Banach theorem:

Theorem 2.4 (Hanh-Banach)**.** *Let $A \subseteq B \subseteq E$ be locally convex spaces. If for every $\varphi \in E'$ such that $\varphi|A = 0$ satisfies $\varphi|B = 0$, then A is dense in B.*

One can prove this as a consequence of the locally convex Hanh-Banach separation theorem (since if it fails, then A is convex and closed in B, and so can be separated). See Lang [Lan93, Appendix IV, Theorem 1.2].

2.2. **Runge exhaustions.** In this section we construct exhaustions of noncompact Riemann surfaces, which have useful connectivity properties, and are well-behaved with respect to Laplace's equation.

Definition 2.5. Let $Y \subseteq X$. The *Runge hull* $h(Y)$ of Y is the union of Y with all precompact components of $X \setminus Y$. A *Runge set* is a set which is equal to its Runge hull.

If Y is an open set, then every component of $X \setminus Y$ is closed. Thus an open set Y is a Runge set iff every component of $X \setminus Y$ is not compact. Therefore the Runge hull of a Runge set Y is Y itself, and $Y \subseteq Z$ implies $h(Y) \subseteq h(Z)$. Let Y be a Runge set. Then if Y is closed or compact, so is $h(Y)$. This is a straightforward exercise in point-set topology.

We now show that any two compact sets with proper containment sandwich a Runge set.

Lemma 2.6. *Let K_1, K_2 be compact subsets of X such that $K_1 \subseteq K_2^o$ and K_2 is Runge. Then there is a Runge open set $Y \subseteq X$ contained in K_2 such that the Dirichlet problem for the Laplace equation on Y is well-posed.*

Proof. For every $x \in \partial K_2$ we can find a compact coordinate disc D centered on x which does not meet K_1. Since ∂K_2 is compact, let $D_0, \dots, D_{m-1}$ be a cover of ∂K_2 by such discs, and let $Y = K_2 \setminus \bigcup_{j<m} D_j$. Then Y is open and contains K_1. By construction, Y meets the hypotheses of Theorem 2.1, so the Dirichlet problem is well-posed.

Since K_2 is Runge, every component of $X \setminus K_2$ is not precompact. On the other hand, every D_j meets a component of $X \setminus K_2$, and is connnected. Therefore no component of $X \setminus Y$ is precompact; therefore Y is Runge. □

Lemma 2.7. *For every Runge open set Y, the components of Y are also Runge open sets.*

Proof. Let Y_i be the components of Y, and let A_i be the components of $A = X \setminus Y$, so that the A_i are closed but not compact. The claim is trivial if A is empty, so assume otherwise. Since X is connected, for every i, $\overline{Y}_i$ meets A.

Now we claim that if C is a component of $X \setminus Y_i$, then C meets A. The only way that this could fail is if $C \subseteq Y$, and thus there is j such that $C \cap Y_j$ is nonempty. But C is closed and Y_j is connected, so $\overline{Y}_j \subseteq C$, and since $\overline{Y}_j$ meets A, so does C.

In particular, C meets some A_k, but C is a component and A is connected, so $A_k \subseteq C$. Since A_k is closed but not compact, C must not be precompact, so Y_i is Runge. □

Theorem 2.8. *Every noncompact Riemann surface X has a Runge open cover (Y_j) such that $Y_j \Subset Y_{j+1}$ and for every j, Y_j is connected and the Dirichlet problem on Y_j is well-posed.*

Proof. Since X is second-countable, it suffices to show that for every compact set $K \subseteq X$ there is a Runge open set $Y \Subset X$ such that $K \subseteq Y$, Y is connected, and the Dirichlet problem on Y is well-posed.

Let K' be a connected compact set containing K, and K'' a compact set such that $K' \subset (K'')^o$. By Lemma 2.6, we can find a Runge open set Y' such that $K' \subseteq Y' \subseteq K''$ and the Dirichlet problem on Y' is well-posed. Let Y be the component of Y' that contains K'; by arbitrarily extending Dirichlet data on $\partial Y'$ to ∂Y, we can solve the Laplace equation on Y. By Lemma 2.7, Y is Runge. □

3. Analysis on noncompact Riemann surfaces

3.1. **Existence of Runge approximations.** Let us now prove the Runge approximation theorem, which says that we can approximate in $\mathscr{O}$ a local holomorphic function by a global holomorphic section, provided that our Riemann surface is not compact.

Lemma 3.1. *Let Z be an open subset of X, $S \in (\mathscr{E}')^{0,1}(X)$, and suppose that for every $g \in \mathscr{D}(Z)$, $\langle S, d''g\rangle = 0$. Then there is $\sigma \in \Omega(X)$ such that for every $\omega \in \mathscr{D}^{0,1}(Z)$,*

$$\langle S, \omega\rangle = \iint_Z \sigma \wedge \omega.$$

Proof. By a partition of unity argument, we may restrict to the domain Y of a coordinate z. For every $f \in \mathscr{D}(Y)$, let $\tilde{f}$ be the $(0,1)$-form $f\ d\overline{z}$, extended to all of X. Then we can define a holomorphic distribution $\tilde{S}$ on Y by $\langle \tilde{S}, f\rangle = \langle S, \tilde{f}\rangle$. In particular, by Weyl's lemma there is $h \in \mathscr{O}(Y)$ such that for every $f \in \mathscr{D}(Y)$,

$$\langle S, \tilde{f}\rangle = \iint_Y h(z) f(z)\ dz \wedge d\overline{z}.$$

Thus we can set $\sigma = h\ dz$. □

Lemma 3.2. *Let Y be a precompact open Runge subset of a noncompact Riemann surface X. Then for every open set Y' such that $Y \subseteq Y' \Subset X$, the image of the natural map $\mathscr{O}(Y') \to \mathscr{O}(Y)$ is dense.*

Proof. Let $\beta : \mathscr{O}(Y') \to \mathscr{O}(Y)$ be the natural map. It suffices by the Hanh-Banach theorem to show that for every compactly supported distribution T on Y, if T annihilates $\beta(\mathscr{O}(Y'))$, then T annihilates $\mathscr{O}(Y)$.

Let

$$V = \{(\omega, f) \in \mathscr{E}^{0,1}(X) \times \mathscr{E}(Y') : d'' f = \omega | Y'\}.$$

Since Y' is precompact, we can invert d'' and so for every ω find f such that $(\omega, f) \in V$. Define a compactly supported distribution $S : \mathscr{E}^{0,1}(X) \to \mathbf{C}$ to make the diagram

$$\begin{array}{ccccc} V & \longrightarrow & \mathscr{E}(Y') & \xrightarrow{\beta} & \mathscr{E}(Y) \\ \big\downarrow & & & & \big\downarrow T \\ \mathscr{E}^{0,1}(X) & & \xrightarrow{\quad S \quad} & & \mathbf{C} \end{array}$$

commute.

Let $K = \operatorname{supp} T$. Then by Lemma 3.1, there is $\sigma \in \Omega(X \setminus K)$ such that

$$\langle S, \omega \rangle = \iint_{X \setminus K} \sigma \wedge \omega$$

whenever $\omega \in \mathscr{E}^{0,1}(X)$ and $\operatorname{supp} \omega \Subset X \setminus K$. If $L = \operatorname{supp} S$, then $\operatorname{supp} \sigma \subseteq K \cup L$.

Since every component of $X \setminus h(K)$ is not precompact, it meets $X \setminus K \cup L$. So $\sigma | X \setminus h(K) = 0$. That is, if $\omega \in \mathscr{E}^{0,1}(X)$ and $\operatorname{supp} \omega \Subset X \setminus h(K)$, then $\langle S, \omega \rangle = 0$.

Let $f \in \mathscr{O}(Y)$. Since Y is Runge, $h(K) \subseteq Y$, so there is $g \in \mathscr{E}(X)$ wth $f = g$ near $h(K)$ and $\operatorname{supp} g \Subset Y$. So

$$\langle T, f \rangle = \langle T, g | Y \rangle = \langle S, d'' g \rangle.$$

Since g is holomorphic near $h(K)$, $\operatorname{supp}(d'' g) \Subset X \setminus h(K)$ so $\langle S, d'' g \rangle = 0$. Thus $\langle T, f \rangle = 0$. ☐

Theorem 3.3 (Runge approximation)**.** *Let X be a noncompact Riemann surface, Y an open set whose complement contains no compact component. Then every holomorphic function on Y can be approximated in $\mathscr{O}(Y)$ by an element of $\mathscr{O}(X)$.*

Proof. Let $f \in \mathscr{O}(Y)$, $K \subset Y$ is compact, and $\varepsilon > 0$; we must find $F \in \mathscr{O}(X)$ with

$$||f - F||_{L^\infty(K)} \lesssim \varepsilon. \tag{1}$$

Since we only care about compact subsets of Y, we might as well assume Y is precompact. Then by Theorem 2.8, we can find a Runge open cover (Y_j) such that $Y \Subset Y_1$ and $Y_j \Subset Y_{j+1}$. By Lemma 3.2 we can find $f_1 \in \mathscr{O}(Y_1)$ with

$$||f_1 - f||_{L^\infty(K)} < \varepsilon.$$

Now by Lemma 3.2 and induction we can find f_n so that

$$||f_n - f_{n-1}||_{L^\infty(K)} \leq ||f_n - f_{n-1}||_{L^\infty(\overline{Y}_{n-2})} < \frac{\varepsilon}{2^n}.$$

Thus there is $F \in \mathscr{O}(X)$ which is the pointwise limit of the f_n, which satisfies (1). □

3.2. **Existence of Weierstrass products.** We now show a version of the Weierstrass products theorem for noncompact Riemann surfaces. This will follow easily once we show that all divisors on a noncompact Riemann surface are linearly equivalent – equivalently, every line bundle on a Riemann surface is trivial. The idea here is that, to show that every line bundle is trivial, we must show $H^1(X, \mathscr{O}^*)$ is trivial – but to do that, we will first need to show $H^1(X, \mathscr{O}) = 0$, and then show that we can take the "logarithm" of a nonnegative cocycle in $Z^1(X, \mathscr{O}^*)$.

Theorem 3.4 (Mittag-Leffler)**.** *Let X be a noncompact Riemann surface. Then $H^1(X, \mathscr{O}) = 0$.*

Proof. Since $H^1(X, \mathscr{O}) = \mathscr{E}^{1,0}(X)/d''\mathscr{E}(X)$, it suffices to show that for every $\omega \in \mathscr{E}^{0,1}(X)$ there is $f \in \mathscr{E}(X)$ such that $d''f = \omega$. To do this, we first note that we can do this in a precompact open set $Y \Subset X$, since d'' is locally invertible.

By Theorem 2.8, we can set $Y_0 = Y$ and choose $Y_j \subseteq Y_{j+1}$ so that (Y_j) is an open cover of X and Y_j is a connected open Runge set whenever $j > 0$. First choose $f_0 \in \mathscr{E}(Y_0)$ so that $d''f_0 = \omega|Y_0$. Given $f_0, \dots, f_n$, set $g_{n+1} \in \mathscr{E}(Y_{n+1})$ to satisfy $d''g_{n+1} = \omega|Y_{n+1}$. Then $g_{n+1}|Y_n - f_n$ is holomorphic, so we can find a Runge approximation $h \in \mathscr{O}(Y_{n+1})$ to $g_{n+1} - f_n$, and then set $f_{n+1} = g_{n+1} - h$. Then $d''f_{n+1} = \omega|Y_{n+1}$ and $||f_{n+1} - f_n||_{L^\infty(Y_{n-1})} < 2^{-n}$. Then the f_n form a Cauchy sequence that must converge to a solution f of $d''f = \omega$. □

Lemma 3.5. *Every divisor on a noncompact Riemann surface has a weak solution.*

Proof. Let D be a divisor. After applying a partition of unity we may assume that D is a single point a_0. By Lemma 3.2 we can find compact Runge sets K_j with $a_0 \notin K_0$, $K_j \subseteq K_{j+1}$, and $\bigcup_j K_j = X$.

So we must show that there is a weak solution φ of a_0 with $\varphi|K_0 = 1$. Indeed, since K_0 is Runge, the component U containing a_0 is not precompact. So there is $a_1 \notin K_1$ and a curve from a_1 to a_0 in U. Iterating we get $a_k \in X \notin K_k$ and curves c_k from a_{k+1} to a_k. In particular, $\partial c_k = a_{k+1} - a_k$, and there are weak solutions φ_k of the divisors ∂c_k which are 1 on K_{j+k}. Then $\varphi = \prod_k \varphi_k$ is a weak solution of a_0. □

Theorem 3.6 (Weierstrass product)**.** *Let X be a noncompact Riemann surface. Then $H^1(X, \mathscr{O}^*) = 0$.*

Proof. Let D be a divisor. We can solve D in any simply connected coordinate chart, so we can choose an open cover (U_i) of simply connected sets such that there are $f_i \in \mathscr{M}^*(U_i)$ with $(f_i) = D|U_i$. Then $f_{ij} = f_i/f_j \in \mathscr{O}^*(U_i \cap U_j)$. Let ψ be a weak solution of D, which exists by Lemma 3.5. Then we can write $\psi|U_i = e^{\varphi_i} f_i$ (since

$\psi|U_i/f_i$ has no zeroes or poles). Then $f_{ij} = e^{\varphi_j - \varphi_i}$, so $\varphi_{ij} = \varphi_i - \varphi_j \in \mathscr{O}(U_i \cap U_j)$. Also $\varphi_{ij} + \varphi_{jk} = \varphi_{ik}$, so $\Phi = (\varphi_{ij})$ is a cocycle for $\mathscr{O}$. Since $H^1(X, \mathscr{O}) = 0$ by Mittag-Leffler's theorem, it follows that Φ is a coboundary, so (f_{ij}) is also a coboundary. Therefore there is a global solution f to D. □

If $H^1(X, \mathscr{O}^*) = 0$ and g is a nonconstant meromorphic function on X, then any solution f to $-(dg)$ defines a holomorphic 1-form $f\ dg$ with no zeroes.

4. Holomorphic de Rham cohomology

We now study properties of the holomorphic de Rham cohomology $H^\bullet_{\mathscr{O}}(X, \mathbf{C})$ of a Riemann surface X, obtaining a form of the Riemann mapping theorem as a consequence. We will then show the uniformization theorem by taking an exhaustion of X by sets isomorphic to $\mathbf{D}$.

If $H^1_{\mathscr{O}}(X, \mathbf{C}) = 0$ then every function in $\mathscr{O}^*(X)$ has a logarithm and a square root. The proofs are as usual. It follows that every harmonic function on X is the real part of a holomorphic function.

Let us first show the version of the Riemann mapping theorem that was known to Riemann:

Lemma 4.1. *Suppose that X is a noncompact Riemann surface, $Y \Subset X$ is a connected open set, $a \in Y$, and $H^1_{\mathscr{O}}(Y, \mathbf{C}) = 0$. If the Dirichlet problem for the Laplace equation on Y is well-posed, then there is an isomorphism $f : Y \to \mathbf{D}$ with $f(a) = 0$.*

Proof. By the Weierstrass product theorem, there is a holomorphic function g on X which is nonzero on $X \setminus a$, with a single zero at a. In particular, $\log |g|$ is continuous on ∂Y, so there is a continuous function $u : \overline{Y} \to \mathbf{R}$ such that $u = \log |g|$ on ∂Y and $\Delta u = 0$ on Y. Then u is the real part of a holomorphic function h on Y. Let $f = e^{-h} g$.

Now we show $f(Y) \subseteq \mathbf{D}$. In fact, if $y \in Y \setminus a$,

$$|f(y)| = e^{-u(y)} |g(y)| = e^{\log |g(y)| - u(y)}.$$

In particular, $\varphi = |f|$ extends continuously to $\overline{Y}$, and $\varphi = 1$ on ∂Y. Thus, by the maximum principle, $|f| < 1$ on Y.

If $r < 1$ and $Y_r = \{|f| \leq r\}$, then Y_r is a closed subset of $\overline{Y}$ and a subset of Y, so that Y_r is compact in Y. Therefore $f : Y \to \mathbf{D}$ is a proper morphism, so f attains each value the same amount of times, but f attains zero exactly once, so f is bijective, and hence an isomorphism. □

We write $\mathbf{D}_r$ for $\{z \in \mathbf{C} : |z| < r\}$. By Cauchy's estimate, we see that if $f : \mathbf{D}_r \to \mathbf{D}_s$ then

$$|f'(0)| \leq \frac{s}{r}.$$

Recall Montel's theorem, which says that a closed and bounded[1] subset of $\mathscr{O}(\mathbf{D})$ is compact. In particular the sets $\{f \in \mathscr{O}(\mathbf{D}) : f(\mathbf{D}) \subseteq \mathbf{D}_r, f(0) = 0\}$ are compact. Now if G is an open subset of $\mathbf{P}^1$ such that $\mathbf{P}^1 \setminus G$ contains an open set, then G can be mapped to a subset of $\mathbf{D}_r$ for some r. Thus the sets

$$\{f \in \mathscr{O}(\mathbf{D}) : f(\mathbf{D}) \subseteq G, f(0) = w\},$$

$w \in G$, are compact.

Let $\mathscr{S}$ be the space of embeddings $F : \mathbf{D} \to \mathbf{C}$ with $F(0) = 0$ and $F'(0) = 1$.

Lemma 4.2. *As a subset of $\mathscr{O}(\mathbf{D})$, $\mathscr{S}$ is compact.*

Proof. Let (f_n) be a sequence in $\mathscr{S}$. Let $r_n > 0$ be the maximum radius such that $\mathbf{D}_{r_n} \subseteq f_n(\mathbf{D})$. The inverse φ_n of f_n maps $\mathbf{D}_{r_n}$ into $\mathbf{D}$, so $1 = \varphi_n'(0) \leq r_n^{-1}$; therefore $r_n \leq 1$. Moreover, by definition of r_n, there is $a_n \in \partial\mathbf{D}_{r_n}$ with $a_n \notin f_n(\mathbf{D})$. So let $g_n = f_n/a_n$; then g_n is an embedding, $\mathbf{D} \subseteq g_n(\mathbf{D})$, and $1 \notin g_n(\mathbf{D})$.

Let $\psi(z)$ be the square root of $z - 1$, chosen so $\psi(0) = i$. Let $U = \psi(\mathbf{D})$. Then, since $g_n(\mathbf{D})$ is isomorphic to $\mathbf{D}$ and so simply connected, and $1 \notin g_n(\mathbf{D})$, ψ extends to $g_n(\mathbf{D})$. Let $h_n = \psi \circ g_n$; then $h_n = \sqrt{g_n - 1}$.

Suppose that $w, -w \in h_n(\mathbf{D})$. Then there are $z_1, z_2 \in \mathbf{D}$ such that $w = h_n(z_1)$ and $-w = h_n(z_2)$. Then $w = w^2$, so $g_n(z_1) = g_n(z_2)$, but g_n is an embedding so $z_1 = z_2$. Therefore $w = -w$ so $w = 0$, and hence $g_n(z_1) = 1$, a contradiction. So if $w \in h_n(\mathbf{D})$ then $-w \notin h_n(\mathbf{D})$.

Since $\mathbf{D} \subseteq g_n(\mathbf{D})$, $U \subseteq h_n(\mathbf{D})$. Thus $-U$ does not meet $h_n(\mathbf{D})$. Since $-U$ is an open set contained in $\mathbf{P}^1 \setminus \bigcup_n h_n(\mathbf{D})$, it follows that the (h_n) have a convergent subsequence. But

$$f_n = a_n(1 + h_n^2)$$

and $|a_n| \leq 1$ is uniformly bounded, so (f_n) has a convergent subsequence, say of limit f.

Finally we show that f is an embedding. If not, then there is $a \in \mathbf{C}$ such that $f - a$ has at least two zeroes. By local stability of zeroes, there are arbitrarily large n such that $f_n - a$ has at least two zeroes, even though f_n is an embedding. So f is an embedding. □

Lemma 4.3. *If Y is a proper open connected subset of $\mathbf{D}$ or $\mathbf{C}$ with $H^1_{\mathscr{O}}(Y, \mathbf{C}) = 0$ then there is $r < 1$ and a holomorphic map $f : Y \to \mathbf{D}_r$ with $f(0) = 0$ and $f'(0) = 1$.*

Proof. Let $a \notin Y$, and let

$$\varphi(z) = \frac{z - a}{1 - \bar{a}z}.$$

[1] A subset of a Fréchet space is said to be bounded if it is bounded in every seminorm.

Then $0 \notin \varphi(Y)$; let g be a square root of $\varphi|Y$. Then $g(Y) \subseteq \mathbf{D}$, so let $b = g(0)$ and

$$\psi(z) = \frac{z-b}{1-\bar{b}z}.$$

Then $h = \psi \circ g : Y \to \mathbf{D}$ satisfies $h(0) = 0$ and

$$h'(0) = \frac{\psi'(b)\varphi'(0)}{2g(0)} = \frac{1-|a|^2}{2b(1-|b|^2)} = \frac{1+|b|^2}{2b}$$

since $b^2 + a = 0$. So $|h'(0)| > 1$, so set $r = 1/|h'(0)|$ and $f = h/h'(0)$. □

5. Proof of Proposition 1.3

By Lemma 2.8, let $Y_n \Subset Y_{n+1}$ be connected Runge open sets with $\bigcup_n Y_n = X$, such that every the Dirichlet problem for the Laplace equation on Y_n is well-posed.

Let ω be a holomorphic 1-form on Y_n. By the Weierstrass product theorem, there is a holomorphic 1-form ω_0 on X with no zeroes, so let $f = \omega/\omega_0$. Let (f_n) be a Runge approximation of f in $\mathscr{O}(X)$. Then if $\alpha \in H_1(Y_n, \mathbf{C})$,

$$\lim_{n\to\infty} \int_\alpha f_n\omega_0 = \int_\alpha \omega.$$

But as $H^1_{\mathscr{O}}(X, \mathbf{C}) = 0$, the left-hand side is zero, so $\int_\alpha \omega = 0$. Therefore $H^1_{\mathscr{O}}(Y_n, \mathbf{C}) = 0$, so by Lemma 4.1, Y_n is isomorphic to $\mathbf{D}$.

Let $a \in Y_0$ and z a coordinate at a. Then there are $r_n > 0$ and isomorphisms $f_n : Y_n \to \mathbf{D}_{r_n}$ such that $f_n(z) = 0$ and $df_n|_{z=0} = dz|_{z=0}$. In particular, $r_n \le r_{n+1}$, since the map $h_n = f_{n+1} \circ f_n^{-1}$ satisfies $h(0) = 0$ and $h'(0) = 1$, and thus $1 = h'(0) \le r_{n+1}/r_n$ by Cauchy's estimate. Let $R = \lim_n r_n$; then we claim that X is isomorphic to $\mathbf{D}_R$.

To accomplish this, we first find a subsequence of the (f_n) which converges in $\mathscr{O}(Y_m)$ for each m. The map $z \mapsto f_0^{-1}(r_0 z)$ is an isomorphism $\mathbf{D} \to Y_0$; set

$$g_n(z) = \frac{1}{r_0} f_n(f_0^{-1}(r_0 z));$$

then $g_n \in \mathscr{S}$, so we can find a subsequence $(f_{n_{0,k}})_k$ which converges in $\mathscr{O}(Y_0)$. Repeating this process with (f_n) replaced by $(f_{n_{j,k}})_k$, we can find a subsequence $(f_{n_{j+1,k}})_k$ which conveges in $\mathscr{O}(Y_{j+1})$ by induction, and then diagonalize to get a subsequence which converges in $\mathscr{O}(Y_m)$ for each m, say to $f \in \mathscr{O}(X)$. Then f is an embedding with $f(a) = 0$ and $df|_{z=0} = dz|_{z=0}$.

Since each of the f_n map into $\mathbf{D}_R$ by definition, so does f. So we just need to show that f is surjective. If not, then $f(X)$ is a proper open connected subset of $\mathbf{D}_R$, so by Lemma 4.3, there is $r < R$ and a holomorphic map $g : f(X) \to \mathbf{D}_r$ with $g(0) = 0$ and $g'(0) = 1$. If n is large enough then $r_n > r$, and then $h = g \circ f \circ f_n^{-1}$ sends $\mathbf{D}_{r_n} \to \mathbf{D}_r$, $h(0) = 0$, and $h'(0) = 1$; but by Cauchy's estimate this implies $r_n \le r$. This is a contradiction, so f is surjective.

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