Formalizations of analysis

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1 Introduction

Despite the role of mathematical analysis as the foundation of physical theories, such as quantum mechanics, the subject is often considered fraught with pathology: the existence of nonmeasurable sets, nowhere continuous functions, and other "monsters." Such complications increase the appeal of constructive mathematics, such as Bishop's attempt at redeveloping analysis from an intuitionistic perspective. We will provide an overview of Bishop's analysis and qualify his claim that classical analysis is an approximation to "constructive truth." Finally, we consider another means of subverting analytic pathology, and contrast it to Bishop's approach.

2 Introducing Bishop's analysis

Prior to the work of E. Bishop [2], mathematical analysis had defied any constructive formalization [12]: unlike algebra and combinatorics, analysis is inherently infinitary in nature, and most intuitionists following L. Brouwer had focused on recursion theory and related fields. Nevertheless, Bishop was able to develop a framework for analysis which recovered even such deep theorems as the Riemann mapping theorem. This lead Bishop to his thesis:

> The extent to which good constructive substitutes exist for the theorems of classical mathematics can be regarded as a demonstration that classical mathematics has a substantial underpinning of constructive truth. [2]

Bishop's proof system, known as **BISH**, is essentially intuitionistic: he begins by redefining the connectives to be as prescribed by the Brouwer-Heyting-Kolmogorov interpretation (BHK), and defines a set to be a collection of objects with an equivalence relation, written $=$. Bishop defines equality this manner to facilitate avoiding the definition of $\neq$ as a negation; Bishop wants to avoid negative notions wherever possible to minimize instances where he may run afoul of the law of the excluded middle (LEM), and so simply defines $\neq$ to be a symmetric relation such that if $x \neq y$ then either $x \neq z$ or $y \neq z$, and such that $(x = y) \wedge (x \neq y)$ proves $(0 = 1)$, in accordance with BHK. (In particular, if x and y are reals, then $x \neq y$ iff $x > y$ or $y > x$.)

As an example, we prove the constructive intermediate value theorem. The Heine-Cantor theorem, stating that a continuous function on a compact set is in fact uniformly so, cannot be proven from **BISH**, as we will discuss further in Sections 3 and 5. Instead, Bishop simply defines a continuous function is equipped by definition with a *modulus of continuity*, a method ω such that for each $\varepsilon > 0$, if $|y - x| < \omega(\varepsilon)$, then $|f(y) - f(x)| < \varepsilon$. Clearly any such function is classically uniformly continuous. Besides this, the Heine-Borel theorem is not constructively valid, because it proves the weak Kőnig's lemma (WKL) [6], so Bishop simply defines a compact interval to be one of the form $[\alpha, \beta]$.

Theorem 1 (intermediate value theorem)**.** *Let I be a compact interval, and let $f : I \to \mathbb{R}$ be continuous. If $a < b$ are points of I, with $f(a) < f(b)$, then for each $y \in [f(a), f(b)]$ there exists $x \in [a, b]$ such that for each $\varepsilon > 0$, $|f(x) - y| < \varepsilon$.*

Proof. Let $y \in [f(a), f(b)]$, and let $\delta = \inf_{x\in[a,b]} |f(x) - y|$.[1] If $\delta > 0$, then $f(a) - y \leq -\delta$ and $f(b) - y \geq \delta$.

Let ω be a modulus of continuity for f, and choose $x_0 \leq x_1 \leq \cdots \leq x_n$ such that $x_0 = a$, $x_n = b$, and for each k, $x_{k+1} - x_k < \omega(\delta)$. Then $|f(x_{k+1}) - y - (f(x_k) - y)| = |f(x_{k+1}) - f(x_k)| < \delta$ but by definition, if $x \in [a, b]$, then $|f(x) - y| \geq \delta$, so either $f(x_k) - y$ and $f(x_{k+1}) - y$ are both positive, or they are both negative. By induction, either $f(a) - y > 0$ and $f(b) - y > 0$ are both positive, or $f(a) - y < 0$ and $f(b) - y < 0$. So we cannot have $\delta > 0$. Therefore, given $\varepsilon > 0$, $\delta < \varepsilon$, which was to be shown. □

Notice that if we know that infinitesimals do not exist, then this theorem immediately implies the classical intermediate value theorem, as then $f(x) = y$. The trouble is that we cannot assert $\delta = 0$ even though we know $(\delta \geq 0) \wedge \neg(\delta > 0)$. This is

[1] **BISH** is not strong enough to prove that the minimum is attained.

essentially an immediate consequence of the rejection of LEM, and in particular the rejection of the *limited principle of omniscience* (LPO) [4]:

Theorem 2 (limited principle of omniscience)**.** *Assume LEM. Let $\{x_n\}$ be a binary sequence. Either for each $n \in \mathbb{N}$, $x_n = 0$, or there is a $n \in \mathbb{N}$ such that $x_n = 1$.*

If LPO was valid in **BISH**, then there would be an algorithm A to determine whether there was some n such that $x_n = 1$; but if so, then A could solve the halting problem, so Bishop explicitly rejects LPO. Stronger, he blames LPO for the failure of classical analysis to be constructive, rather than the usual scapegoat, claiming that "Applications of the axiom of choice in classical mathematics either are irrelevant or are combined with a sweeping use of the principle of omniscience." [2] But with the failure of LPO, if $\delta \geq 0$, then $\neg\neg(\delta = 0) = \forall \varepsilon > 0\ \delta < \varepsilon$.

Several classical theorems of analysis are constructivized without a hitch, including the Riemann mapping theorem, the Gram-Schmidt algorithm, and the Fourier inversion formula on locally compact abelian groups. Still others, such as the Hanh-Banach separation theorem, the Banach-Alaglou theorem, and the Tietze extension theorem, must be only weakened slightly [2]. This is especially remarkable, because the Banach-Alaglou theorem is usually taught as a corollary of Tychnoff's theorem (which is known to prove the axiom of choice) [9], and gives credence to Bishop's thesis.

3 Conflicts of continuity

Though Bishop's work is frequently compared to that of Brouwer, **BISH** is compatible with classical mathematics, say **ZFC**, while Brouwer's theory, **INT**, includes the axiom of continuous choice (ACC):

Axiom 3 (continuous choice)**.** *Let us work in* **INT**. *Then every function $\mathbb{N}^\mathbb{N} \to \mathbb{N}$ is continuous, and for each $P \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}$ modeling $\forall x \in \mathbb{N}^\mathbb{N}\ \exists y \in \mathbb{N} (x, y) \in P$, then there exists a function $\mathbb{N}^\mathbb{N} \to \mathbb{N}$ whose graph is P.*

INT is the theory **BISH** + ACC [4].

Brouwer even considered ACC a theorem, giving the following argument. First, **BISH** cannot prove the existence of a discontinuous function $\mathbb{N}^\mathbb{N} \to \mathbb{N}$ – and so if one denies the existence of anything which he cannot find an example of, it will follow that all functions on $\mathbb{N}^\mathbb{N}$ are continuous. Second, the existence of a choice function can be expressed as the schema φ_P, ranging over $P \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}$, given by

$$\varphi_P = ((\forall x \in \mathbb{N}^\mathbb{N}\ \forall y \in \mathbb{N}\ (x, y) \in P) \implies (\exists f : X \to Y\ \forall x \in \mathbb{N}^\mathbb{N}\ (x, f(x)) \in P)).$$

In order to give a BHK proof of φ_P, we must provide a method which converts proofs of $\psi_P = (\forall x \in \mathbb{N}^\mathbb{N}\ \exists y \in \mathbb{N}(x, y) \in P)$ into functions f and proofs that $\forall x \in \mathbb{N}^\mathbb{N}\ (x, f(x)) \in P))$. But if $M \vdash \psi_P$, according to BHK, then M is a method which takes x and returns $M_1(x) \in \mathbb{N}$ and $M_2(x) \vdash ((x, M_1(x)) \in P)$, so if we conflate the notions of method and function, it will follow that $f = M_1$ [11]. In any case, ACC contradicts LPO, which is enough to prove the existence of a discontinuous function on Baire space [4].

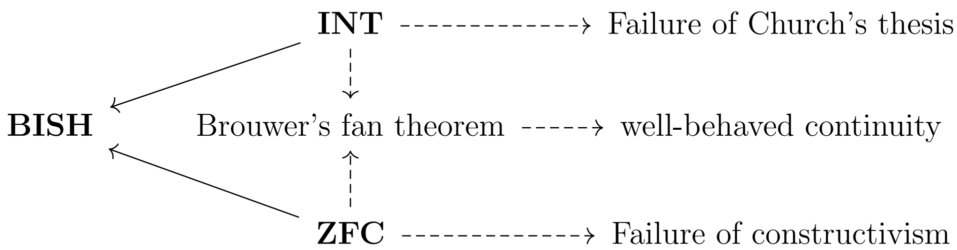
However, **BISH** does not take ACC as an axiom, since, as Bishop puts it, "to accept Brouwer's arguments as a proof would destroy the character of mathematics." Worse, if one extends Brouwer's argument to *arbitrary* relations P rather than those on $\mathbb{N}^\mathbb{N} \times \mathbb{N}$, one proves the axiom of choice, which proves LPO over **BISH**, and thus achieves a contradiction. Perhaps Brouwer's argument is more effective as a case against BHK, which is deliberately vague about what a "method" is, than as a "proof" of ACC.

ACC is enough to prove a classical result, Brouwer's fan theorem (FAN) [4]:

Theorem 4 (Brouwer's fan theorem)**.** *Let us suppose ACC or LEM + WKL. Let S be a set of finite binary sequences such that each $x \in 2^\mathbb{N}$ has an initial segment $x^* \in S$. Then there exists $N \in \mathbb{N}$ such that if $x \in 2^\mathbb{N}$ then x^* can be chosen to have length at most N.*

FAN, in turn, is used in several proofs that depend on continuity or compactness, including the Heine-Cantor theorem, and that the composite of continuous functions is continuous [5]. All of these theorems are left undecided by **BISH**, though Bishop considered them true. For more consequences of FAN, we refer the reader to [14].

It appears curious that Bishop would develop an axiomatic system which cannot prove many statements which he considered true – in fact, one could even consider **BISH** as a framework from which one could develop constructive theories, but not a useful theory in itself. Nevertheless, the failings of **BISH** are inevitable, as the intuitionistic proof of FAN is incompatible with Church's thesis [10]. A depiction of the relationship between the theories discussed is given below.



That one must give up either FAN, constructivism, or Church's thesis is a significant undermining of Bishop's thesis. While constructive and classical mathematics are similar, they cannot be too similar: either one sacrifices recursion theory to better recover analysis, or one sacrifices part of analysis to protect recursion theory. That whether one goes full-in with constructivism (**INT**) or accepts classicism (**ZFC**), one ends up with FAN and its menagerie of consequences, is strong evidence that FAN is true in a Platonic sense; but Church's thesis is empirically true.

4 Ontological and epistemic concerns

Recall that in the statement of Theorem 1, we could not rule out the existence of infinitesimals. If we accept LPO, since elements of $[0, 1]$ can be coded as binary sequences simply by working in base 2, $\text{LPO} \vdash ((\delta \geq 0) \wedge \neg(\delta > 0)) \implies \delta = 0$. But let ζ denote the Riemann zeta function, and let $x_n = 0$ provided that $\zeta(1/2 + in) = 0$; otherwise let $x_n = 1$. Then $\{x_n\}$ codes a real $x \in [0, 1]$; in particular, if there exists a proof of the Riemann hypothesis, then $x = 0$. But no counterexample is known, so in the intuitionistic ontology, it is not the case that there is $n \in \mathbb{N}$ with $x_n = 1$, and it follows that for each $\varepsilon > 0$ we can prove that $x < \varepsilon$, even though we do not have $x = 0$. In particular, $\{x_n\}$ is a counterexample to LPO, what Brouwer calls a *fugitive sequence* [2].

Fugitives are scarcely limited to analysis; rather, they are built into the philosophy of intuitionism. Heyting, for one, considers the example where ℓ is the greatest twin prime, or $\ell = 1$ if there does not exist a twin prime. However, Heyting denies that ℓ is even a well-defined number, while Bishop is willing to accept that $x \in \mathbb{R}$ [1].

Either way, a truth value changes – that ℓ is well-defined, or that x is infinitesimal – when one resolves either the twin prime conjecture or Riemann hypothesis.

Heyting defends this by arguing that to assert that ℓ was well-defined or that x was infinitesimal prior to a proof of the conjecture is to make a *metaphysical* assertion about the nature of mathematics; to argue that ℓ was well-defined implies that in some Platonic sense, the behavior of ℓ is fixed.

But let us suppose that the twin prime conjecture is decidable in our favorite formal system (whether that is **ZFC**, **BISH**, or **INT**); since the twin prime conjecture is ultimately a statement about finite objects, this is a reasonable assumption. If a mathematician is ever able to resolve the twin prime conjecture, then he will get the same truth value as another mathematician working independently; and so the value of ℓ has already been determined, and it was that it was not known. A sufficiently advanced alien society from Alpha Centauri would arrive at the same value of ℓ – and thus the value of ℓ must somehow exist outside of human thought.

In this sense, to say that ℓ was not well-defined because we did not know to compute the value is the same as to claim that, for each possible origin of life φ, φ was not the origin of life or else φ was not-not the origin of life, since we as a society do not *know* that φ was not the origin of life, and just as we cannot check each possible twin prime and therefore must give a proof by other means, we cannot travel back in time to verify the truth value of φ; yet this conclusion is absurd.

Classical and constructive mathematics therefore diverge sharply on the nature of fugitives, further undermining Bishop's thesis. However, the damage done to actual theorems is minimal, and usually manifests in the same form as in Theorem 1: morally, the same result holds (intervals are sent to intervals), but there is a slight caveat to allow for fugitive behavior (there may be an infinitesimal gap in the image interval). If one is therefore willing to turn a blind eye to the philosophically curious origins of such caveats, constructive and classical mathematics remain effective approximations to each other.

5 The pointless alternative

Let us now consider another possible constructivisation of analysis, which avoids analytic pathology by sacrificing the primacy of the notion of a point, rather than FAN or Church's thesis. As noted by Picado and Pultr [8], points are unrealistic models for computation, as it is impossible to make measurements with absolute precision, and so functions defined by their actions on points is perhaps inappropriate for use in physics and engineering. In pointless topology, however, open sets are taken as primitive, viewed as elements of a join-complete Heyting algebra (or, in brief, a *frame*).

Let X be a (T_0) topological space, and ΩX be its topology. If $\mathcal{F}$ is a proper filter in a lattice L, we say that $\mathcal{F}$ is *completely join-irreducible* if, whenever $\bigvee \mathcal{A} \in \mathcal{F}$, $\mathcal{A} \cap \mathcal{F}$ is nonempty. On the other hand, if $\mathcal{F}$ is a proper filter in ΩX such that, for some $x \in X$, $\mathcal{F} = \{U \in \Omega X : x \in U\}$, we say that $\mathcal{F}$ is *principal*. If a proper filter in ΩX is completely join-irreducible if and only if it is principal, we will say that X is a *sober space*. Since every point $x \in X$ is contained in a principal ultrafilter, we can identify x with some completely join-irreducible filter provided that X is sober. All Hausdorff spaces, and therefore the vast majority of topological spaces encountered in analysis, are sober.

If L is a frame, we let $\mathfrak{F}L$ denote the set of all completely join-irreducible filters in L. In fact $\mathfrak{F}L$ is a topological space, with open sets of the form $a^* = \{\mathcal{F} \in \mathfrak{F}(L) : a \in \mathcal{F}\}$, for $a \in L$. Thus we can identify a with a^*, and consider L the topology on $\mathfrak{F}L$. If M is also a frame, we define morphisms $f : L \to M$ to be lattice morphisms such that $f(\bigvee A) = \bigvee f(A)$ for all $A \subseteq L$.

Thus we arrive at a duality between frames and sober spaces:

Theorem 5 (pointless duality)**.** *The map Ω is a fully faithful, contravariant functor from sober spaces to frames, and $\mathfrak{F}\Omega$ is the identity functor.*

We omit the proof, but refer the reader to [8].

Thus we are justified in *defining* a topological space to be a frame, and defining a continuous mapping to be an opposite morphism. One can define uniform structures and metrics in frames, recovering the Heine-Cantor theorem. With this so-called "pointless" framework, many classical theorems which are highly nonconstructive, such as those of Stone-Čech and Tychonoff, have been given proofs which avoid the axiom of choice and LEM. However, the topological spaces constructed in such proofs are not guaranteed to have any points without an appeal to the Boolean prime ideal theorem or worse [3].

A similar approach can be taken to recover a pointless measure theory [13], which perhaps perhaps better models how analysts actually think of measure theory, by assuming that all functions are measurable and identifying null sets with the empty set (and thus identifying functions which disagree on null sets), as seen in, e.g., Evans [7].

The successes of pointless measure theory and topology lend credence to Bishop's claim that classical mathematics approximates constructive mathematics. However, the pointless approach provides a significant algebraization and simplification, of various classical theorems of topology and analysis, contrasted to the complications of Bishop's formalization; and besides contradicts Bishop's assertion that "[v]ery little is left of general topology after that vehicle of classical mathematics has been taken apart and reassembled constructively."

6 Conclusion

Both through **BISH** and the pointless approach, one can constructively recover analysis to a close approximation of the classical theory. Certainly such a formalism is mathematically valid, and at times can even simply proofs and algorithms. In that sense, Bishop's original goal can be seen as an outright success.

However, while one does not lose much of mathematics in passing to a constructive setting, one does face philosophical difficulties, in that one can argue for the existence of fugitive reals and the failure of Church's thesis, as seems to be inevitable in the intuitionistic setting. These critiques are not so much due to Bishop's failings as they are to the philosophy of Brouwer and Heyting in general, but they still undermine Bishop's thesis. Most ironically, the field that Bishop considered most hopeless to constructivize – general topology – is the one that suffers the fewest complications when passing to the intuitionistic ontology.

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