Research Statement

My research area is geometric measure theory (GMT), which is concerned with geometric structures of very low regularity, and calculus of variations, which studies partial differential equations (PDE) as giving solutions to suitable infinite-dimensional optimization problems. GMT and the calculus of variations have been closely linked since seminal work on the area-minimizing surfaces: it is much easier to construct such a surface $N \subset \mathbb{R}^3$ in a generalized low-regularity sense than to show that N is actually smooth [Giu84].

My research has focused variational problems in the function spaces BV and L^∞. The associated PDE for these problems are highly degenerate, and one is limited in what classical PDE techniques can be applied to them. The first example of a variational problem in L^∞ is, given a family of maps between two metric spaces, to find a map which minimizes its Lipschitz constant. In the 1980s, Thurston [Thu98] observed that the Lipschitz minimization problem between hyperbolic surfaces has an integral role in Teichmüller theory, the study of deformation of Riemann surfaces. However, it was not until recent work of Daskalopoulos and Uhlenbeck [DU22b; DU22a; DU24] that L^∞-variational techniques were brought to bear on Teichmüller theory.

In my thesis [Bac24b; Bac24a; Bac24c], I emphasize the role of the duality between BV and L^∞ problems, which was previously studied by Górny and Mazón [GM22] and by Daskalopoulos and Uhlenbeck [DU22b]. Solutions of BV problems can be profitably viewed as a generalization of area-minimizing submanifolds. Dually, solutions of L^∞ problems are a generalization of *calibrations*: certificates that some family of area-minimizing submanifolds are actually area-minimizing. Among the geometric applications of the BV-L^∞ duality, I studied the *stable norm*, a natural norm on the homology group $H_{d-1}(M, \mathbb{R})$ of a closed oriented Riemannian manifold M, and showed that the behavior of the stable norm is closely analogous to that of the earthquake norm in Teichmüller theory. On the analytic side, I showed that while a complete understanding of L^∞ variational systems (as opposed to the scalar case) remains far out of reach, one can obtain partial results by passing to the dual BV problem.

More broadly, I am interested in applications of geometric measure theory throughout mathematics. I have established a case of the *fractal uncertainty principle*, which asserts that a function and its Fourier transform cannot both concentrate near sets of fractalline geometry. I am also interested in the question of whether the connections between GMT and computability theory have consequences in the calculus of variations.

Functions of least gradient

A function u on a domain U is said to have *least gradient* if it minimizes its BV seminorm, $\int_U |\nabla u(x)| \, dx$, among all functions meeting a boundary condition. Thus, the problem of finding a function of least gradient subject to a boundary condition is the first example of a variational problem in BV. BV problems have close connections to optimal transport, image processing, and an inverse problem for magnetic resonance imaging (MRI) [TT19]. However, the least gradient problem was originally studied by Miranda [Mir65] for its application to area-minimizing hypersurfaces, as we now explain.

By the coarea formula, the level sets $\partial\{u > t\}$ of a function of least gradient are area-minimizing hypersurfaces. In particular, if $2 \leq d \leq 7$, the level sets are smooth manifolds, which locally are graphs of functions which solve the minimal surface equation, an elliptic PDE [Giu84]. So one expects to show that the level sets enjoy more structure than the level sets of a typical BV function.

A *lamination* λ of codimension k in a manifold is a nonempty closed set which admits a partition into submanifolds of codimension k, and such that, locally, one can make a Lipschitz change of coordinates so that λ takes the form $K \times B$, where K is a closed subset of $\mathbb{R}^k$ and B is a ball in $\mathbb{R}^{d-k}$. (Thus, if K is a closed ball in $\mathbb{R}^k$, λ is actually a foliation.) The fibers $\{k\} \times B$ are called *leaves*.

It was observed by Daskalopolous and Uhlenbeck [DU22b] that natural examples of functions of least gradient seem to have level sets which form a lamination. In fact, we have:

Theorem 1 ([Bac24b]). *Assume that $2 \leq d \leq 7$. Then a nonconstant function u of bounded variation on U has least gradient if and only if the level sets of u form a lamination of area-minimizing hypersurfaces of uniformly bounded extrinsic curvature.*

The main technical tool in establishing Theorem 1 is a general result about the existence of minimal laminations, which itself is a consequence of the Harnack inequality for the minimal surface equation, and generalizes a theorem of Solomon [Sol86] on the regularity of foliations.

Theorem 2 ([Bac24b]). *Let S be a set of disjoint minimal hypersurfaces of bounded curvature, whose union is a closed set. Then S is the set of leaves of a lamination.*

As another application of Theorem 2, I obtained implications between, and compactness results for, different modes of convergence on the space of minimal laminations. These generalize results of Thurston [Thu23] in the setting of geodesic laminations on hyperbolic surfaces, and have found application in the Teichmüller theory developed by Daskalopoulos and Uhlenbeck [DU24], as discussed below.

L^∞ variational problems

Extremal Lipschitz maps and tight forms

The first example of a variational problem in L^∞ is the problem of minimizing the Lipschitz constant of a function $u : M \to \mathbb{R}$ (subject to a boundary condition). The solution to this problem is highly nonunique, and so it is better to consider the ∞-Laplacian $\langle \nabla^2 u, \nabla u \otimes \nabla u\rangle = 0$ whose solutions minimize their Lipschitz constant in every open set [Aro67]. This PDE cannot be written in divergence form and its solutions are not C^2; therefore one can only really talk about viscosity solutions of the ∞-Laplacian. The solutions of this PDE, the ∞-*harmonic functions*, are the value functions of a certain stochastic game and so the extremal Lipschitz problem is also of probabilistic and economic interest [Per+11].

Both because of intrinsic interest and applications to Teichmüller theory (see below), we would like to understand the ∞-Laplacian for *maps* $M \to N$. But now we have a fundamental difficulty: viscosity solutions are based on the order structure of the target, so they only make sense when $N = \mathbb{R}$.

Therefore a new approach is needed, and a natural try is to exploit the fact that ∞-harmonic functions are limits of p-harmonic functions as $p \to \infty$. Daskalopoulos and Uhlenbeck [DU22a] proposed to study the extremal Lipschitz problem between manifolds by studying certain anisotropic p-Laplacian systems and taking the limit $p \to \infty$ to obtain ∞-*harmonic maps*. A key feature of this theory is that the convex dual problem to the p-Laplacian converges to a variational problem in

BV, and I propose that we can make some progress in understanding L^∞ variational systems by playing them off against their dual BV systems.

As a warm-up before studying ∞-harmonic maps, I studied *tight forms*, differential $d-1$-forms which are limits as $p \to \infty$ of the PDE

$$dF = 0, \quad d \star (|F|^{p-2}F) = 0.$$

Tight forms are closed forms which minimize their L^∞ norm in their cohomology class, and if $d = 2$ then they are exactly the derivatives of ∞-harmonic functions. At this time the dual BV problem to ∞-harmonic maps is poorly understood, but the dual problem of tight forms is the least gradient problem.

Let M be a closed oriented Riemannian manifold of dimension d. A *calibration* of a hypersurface $N \subset M$ is a closed $d-1$-form F such that $\|F\|_{L^\infty} = 1$ and F pulls back to the area form on N. If F is a calibration of N, then F minimizes its L^∞ norm in its cohomology class and N minimizes its area. The natural norm on $H^{d-1}(M,\mathbb{R})$ is the *costable norm* $\|\cdot\|_\infty$, which is the dual to the *stable norm* $\|\cdot\|_1$ on $H_{d-1}(M,\mathbb{R})$: if $\alpha \in H_{d-1}(M,\mathbb{R})$ contains a closed area-minimizing hypersurface N then $\|\alpha\|_1$ is the area of N. Building on ideas of [MRL14; DU22b], I used Theorem 1 to prove:

Theorem 3 ([Bac24a])**.** *Let M be a closed oriented Riemannian manifold of dimension $2 \le d \le 7$. For every cohomology class $\rho \in H^{d-1}(M,\mathbb{R})$ of costable norm 1 there exists a tight form F representing ρ, and a measured lamination λ of minimal hypersurfaces such that λ is calibrated by F.*

Under very strong regularity assumptions, tight forms minimize their L^∞ norm in every open set, generalizing the minimization properties of ∞-harmonic functions. But the techniques currently available to us do not seem able to prove this in general.

Nevertheless, Theorem 3 suggests a path forwards for the study of ∞-harmonic maps, which I am currently working towards: I propose to develop the theory of the dual BV problem to the extremal Lipschitz problem, including an analogue of Theorem 1. The dual BV problem can be interpreted in terms of vector-valued optimal transport and so probabilistic techniques may be useful. A similar approach is being explored by Katzourakis and Moser [KM24].

Here is a negative result: Let $u : M \to N$ minimize its Lipschitz constant and let v be a solution of the dual BV problem. Then ∇v is supported on a subset of the *stretch set* of u, the set of x such that the operator norm $|\nabla v|_\infty$ attains its maximum at x. (In the application to Teichmüller theory discussed below, the stretch set contains the canonical maximally-stretched lamination.) We would thus like to obtain constraints on the geometry of the stretch set, but Ze-An and I have shown that essentially no constraints exist: if S_u denotes the stretch set of u, u is an extremal Lipschitz map, and $S \supseteq S_u$ is any closed set, then there is an extremal Lipschitz map w such that $S = S_w$. Stronger, we have:

Theorem 4 ([BZA24])**.** *Let M, N be complete Riemannian manifolds, $u : M \to N$ be a Lipschitz map, $K \subseteq M$ a closed set, and $\psi : M \setminus K \to \mathbb{R}_+$ a continuous function such that $\psi \ge |\nabla u|_\infty$. Then there exists a Lipschitz map w homotopic to u relative to K such that for every $x \in M \setminus K$, $\psi(x) = |\nabla w(x)|_\infty$.*

Teichmüller theory and the stable norm

Teichmüller space is the universal cover $\widetilde{\mathcal{M}_g}$ of the moduli space of closed hyperbolic surfaces of genus g. Thurston [Thu98] observed that one can place a metric on $\widetilde{\mathcal{M}_g}$ by considering a

Lipschitz map $f : M \to N$ for any $M, N \in \widetilde{\mathcal{M}_g}$, which minimizes its Lipschitz constant $L(M, N)$ subject to the constraint that f is homotopic to the identity. Then $\log L(M, N)$, the *Thurston stretch distance*, is an asymmetric distance function on $\widetilde{\mathcal{M}_g}$. Thurston also observed that $L(M, N)$ is the maximal factor by which any geodesic can be stretched by every minimizing Lipschitz map homotopic to the identity, and he predicted that one could establish this duality by "convexity ... and the max flow min cut principle." Daskalopoulos and Uhlenbeck [DU24] have carried out this prediction using their ∞-harmonic maps and the dual BV problem. Central to the above duality is the *canonical maximally-stretched lamination*: the smallest geodesic lamination λ in M such that every leaf of λ is stretched by every minimizing Lipschitz map by a factor of $L(M, N)$.

The space of homotopy classes of maps is rather "nonabelian" in that it is determined by homomorphisms $\pi_1(M) \to SO(2, 1)$. As above, we can try to predict properties of the canonical lamination by "abelianizing" the problem, and a natural way to do this is to study functions $u : \widetilde{M} \to \mathbb{R}$ such that du descends to a closed 1-form on M (since the cohomology group $H^1(M, \mathbb{R})$ parametrizes homomorphisms $\pi_1(M) \to \mathbb{R}$).

We work more generally, on a closed oriented Riemannian manifold M of dimension $2 \le d \le 7$. Though the proofs never appeared, it was observed by Auer and Bangert [AB01] that the geometry of the stable norm ball is intimately related to the intersection theory of, and the structure of measured laminations on, M. In [Bac24c] I used Theorem 3 to show that for any $\rho \in H^{d-1}(M, \mathbb{R})$ such that $\|\rho\|_\infty = 1$, there is a (unique, nonempty) lamination λ_ρ of minimal hypersurfaces in M, such that an immersed hypersurface $N \subset M$ is a leaf of λ_ρ iff every $d-1$-form F in ρ with $\|F\|_{L^\infty} = 1$ calibrates N: this lamination is the *canonical calibrated lamination* of ρ. Furthermore, the dual set $\rho^* := \{\alpha \in H_{d-1} : \|\alpha\|_1 = \langle \rho, \alpha \rangle = 1\}$, which is a flat subset of the stable unit ball, is exactly the set of transverse probability measures to λ_ρ.

Theorem 5 ([Bac24c])**.** *For every ρ with $\|\rho\|_\infty = 1$, ρ^* is a compact convex polytope, whose vertices correspond to ergodic measures on λ_ρ, and whose rational vertices correspond to closed leaves of λ_ρ.*

A remarkably similar result holds for the unit ball B of the tangent space to $\widetilde{\mathcal{M}_g}$ at a surface M under its stretch norm: Thurston [Thu98] showed that for any $\rho \in \partial B$, the dual flat ρ^* is a compact convex polytope which encodes information about geodesic laminations on M. As a consequence of Theorem 5, we obtain some results about the stable norm which were predicted by [AB01]:

Theorem 6 ([Bac24c; AB01])**.** *Let $(\Gamma^{(n)})$ be the derived series of $\pi_1(M)$. If the intersection product on $H_{d-1}(M, \mathbb{R})$ has trivial kernel, or $\Gamma^{(1)}/\Gamma^{(2)}$ is a torsion group, then the stable unit ball of $H_{d-1}(M, \mathbb{R})$ is strictly convex.*

In view of the close analogy between Theorem 5 and [Thu98], and the fact that the dual norm to the stretch norm is the so-called earthquake norm, it seems likely that one can relate the convexity properties of the stretch norm at a hyperbolic surface M to the structure of earthquakes at M. Here, an *earthquake* at M is a map $M \to N$ defined by cutting M along a geodesic lamination, twisting, and regluing. Daskalopoulos and Uhlenbeck [DU24] have shown a correspondence between earthquakes at M and $so(2, 1)$-valued measured geodesic laminations on M, and it would be interesting to prove a "nonabelian" analogue of Theorem 6 in the setting of $so(2, 1)$-valued measured laminations and the stretch norm.

Other applications of GMT

The fractal uncertainty principle

Let M be a hyperbolic manifold, and let $\mathcal{L}(M)$ be the multiset of lengths of primitive geodesics of M. The *Selberg zeta function* of M is the analytic continuation of

$$\zeta_M(s) := \prod_{\ell \in \mathcal{L}(M)} \prod_{k=0}^{\infty} (1 - e^{-(s+k)\ell}).$$

As with any zeta function, ζ_M has diophantine applications and the key question is the location of its zeroes [BGS11]. Leng, Tao, and I showed that the zeroes cannot accumulate on near the boundary of the critical strip:

Theorem 7 ([BLT24; DZ16]). *Let M be a convex cocompact hyperbolic manifold such that $\pi_1(M)$ is a Zariski dense subgroup of the isometry group of $\widetilde{M}$. Let δ be the Hausdorff dimension of the limit set of $\pi_1(M)$. Then there exists an explicit $\varepsilon_0 > 0$ such that the Selberg zeta function ζ_M has only finitely many zeroes and poles with real part $\geq \delta - \varepsilon_0$.*

Theorem 7 is one of several results in quantum chaos that was shown by Dyatlov and Zahl [DZ16] to be a consequence of the *fractal uncertainty principle* (*FUP*). Let X, Y be compact subsets of $\mathbb{R}^d$, which we assume have "fractalline geometry", and let X_h, Y_h be their h-neighborhoods. (In Theorem 7, X and Y will be stereographic projections of the limit set of $\pi_1(M)$.) Normalize the Fourier transform as

$$\mathcal{F}_h f(\xi) = \frac{1}{(2\pi h)^{d/2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi/h} f(x)\, dx.$$

If $\delta(X), \delta(Y)$ denote the Hausdorff dimensions of X, Y, and X, Y are Ahlfors-David fractals, then by Hölder's inequality,

$$\|1_{X_h} \mathcal{F}_h 1_{Y_h}\|_{L^2 \to L^2} \lesssim h^\beta \tag{1}$$

where $\beta \geq 1$ and $2\beta \geq d - \delta(X) - \delta(Y)$; FUP asserts that under suitable geometric hypotheses we can improve on this bound.

Outside of special cases, prior to [BLT24] and the work of Cohen [Coh23] which addresses the case $\delta(X) + \delta(Y) \geq d$, FUP was only known when $d = 1$, which in Theorem 7 corresponds to $\dim M = 2$. FUP is not true for general Ahlfors-David fractals when $d \geq 2$. Leng, Tao, and I isolated a quantitative condition known as *nonorthogonality*, which corresponds to Zariski density, and which implies FUP when $\delta(X) + \delta(Y) \leq d$.

Theorem 8 ([BLT24]). *Let X, Y be compact subsets of $\mathbb{R}^d$, which are nonorthogonal to each other, with doubling Hausdorff measures. Then there exists $\varepsilon_0 > 0$ which is an explicit function of the doubling constants of the Hausdorff measures of X, Y, and the nonorthogonality constant of (X, Y), such that (1) holds with $2\beta \geq d - \delta(X) - \delta(Y) + \varepsilon_0$.*

FUP when $d = 1$ and $\delta(X) + \delta(Y) \leq 1$ was proven by Dyatlov and Jin [DJ18]. Recalling that one can realize the standard Cantor set as the space of paths through the full infinite binary tree, the key improvement over [DJ18] is an inductive procedure for recovering trees $T(X), T(Y)$ whose associated "Cantor sets" are X, Y. This construction is rather geometrically involved, as $(T(X), T(Y))$ must satisfy a combinatorial analogue of the nonorthogonality of (X, Y).

In most applications of Theorem 8, including Theorem 7, one is actually interested in a modification of $\mathcal{F}_h$ which has "curvature" in a suitable sense. We currently believe that one can remove the nonorthogonality assumption when $\mathcal{F}_h$ is curved (and so generalize Theorem 7 to all convex cocompact hyperbolic manifolds), and this is our next goal.

Computability and variational problems

Over the past decade, a deep connection between GMT and *computability theory*, the study of the relative complexities of real numbers, has emerged: the Hausdorff dimension of a set $E \subseteq \mathbb{R}^d$ is also the "information density" needed to describe a point $x \in \mathbb{R}^d$ given that $x \in E$ [LL17]. The use of computability theory has led to a new proof of the Kakeya conjecture when $d = 2$ [LL17] and progress on Falconer's distance problem [FS23]. Given GMT's many applications to the calculus of variations, I therefore speculate that computability-theoretic methods shall be useful to solve variational problems. I am part of a joint team of geometric measure theorists and computability theorists investigating this possibility. At this stage we are laying the foundations by studying computability-theoretic properties of rectifiable sets; for example, we can show that if μ is a rectifiable measure and x is a point which is "sufficiently algorithmically complicated" relative to μ, then μ must have a tangent space at x [Bac+24]; this is a significant strengthening of the classical result that for μ-almost every x, μ has a tangent space at x.

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