SOME INTERESTING OPEN PROBLEMS

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1. Calculus of variations in nonreflexive Banach spaces

Problem 1.1 (Evans)**.** Let u be an ∞-harmonic function on a domain in $\mathbf{R}^2$. Must $u \in C^{4/3}_{\text{loc}}$?

Problem 1.2. Consider the scalar PDE

$$A^{ijk\ell}\partial_i\partial_j u \partial_k u \partial_\ell u = F(u)$$

on a domain in $\mathbf{R}^2$, where $F(u)$ is a smooth first-order nonlinearity and A is a smooth elliptic tensor. (Here, you can be loose about what "elliptic tensor" means, but at least the class of PDE considered should include the ∞-Laplacian in the presence of a Riemannian metric.)

Show that the *Evans-Savin theorem* [ES08] holds: there exists $\alpha > 0$ such that $u \in C^{1+\alpha}_{\text{loc}}$.

Problem 1.3. Let u be a function of least gradient on a simply connected domain in $\mathbf{R}^d$. Show that there exist functions u_{ac}, u_C, u_j of least gradient such that:

(1) $u = u_{ac} + u_C + u_j$.
(2) $u_{ac} \in W^{1,1}_{\text{loc}} \cap C^0_{\text{loc}}$.
(3) $u_C \in C^0_{\text{loc}}$ and $\operatorname{supp}(|\,\mathrm{d}u_C|)$ has Lebesgue measure zero.
(4) $\dim_{\mathcal{H}}(\operatorname{supp}|\,\mathrm{d}u_j|) \leq d-1$.

This holds for $d \leq 7$ by [Bac24a, Proposition 4.6]; the obstruction to this holding for arbitrary BV functions is basically topological in nature and so should not be detected by codimension 8 singularities.

Problem 1.4. Let u be a solution of the total variation flow on a convex domain in $\mathbf{R}^d$, with Dirichlet boundary data.

(1) Under what conditions on the initial and boundary data do the level sets of u undergo mean curvature flow?
(2) If the level sets are undergoing mean curvature flow, do they form a lamination?

A Borel set Λ is *Hausdorff equidimensional* if for every $s < \dim_{\mathcal{H}}(\Lambda)$, $x \in \Lambda$, and $r > 0$, the restriction of the Hausdorff measure $\mathcal{H}^s$ to $B(x,r) \cap \Lambda$ is not σ-finite.

Problem 1.5. For each $2 \leq p < \infty$, let u_p be a p-harmonic function on a simply connected, nonpositively curved surface M, all of which have the same boundary data. Let

$$\mathrm{d}v_q := |\,\mathrm{d}u_p|^{p-2} \star \mathrm{d}u_p$$

where $1/p + 1/q = 1$. Then v_q is a well-defined function and there exists a function v of least gradient such that $v_q \to v$ in $L^{3/2}_{\text{loc}}$ along a subsequence. Thus the energy density of u_p concentrates as $p \to \infty$ on the set $\Lambda := \operatorname{supp}(|\,\mathrm{d}v|)$.

Show that Λ is Hausdorff equidimensional. The intuition for this comes from [Bac24b, Theorem 1.6], which is a quantitative version of this in some special cases.

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Given a matrix A, let $Q(A) := (AA^\dagger)^{1/2}$ be the positive-semidefinite part of A. A map $u : M \to N$ is *Schatten p-harmonic* if $u \in W^{1,p}_{\mathrm{loc}}$ and

$$\nabla_u^*(Q(\mathrm{d}u)^{p-2}\,\mathrm{d}u) = 0,$$

where ∇_u^* is the covariant divergence.

Problem 1.6 (Daskalopoulos–Uhlenbeck)**.** Let $u : M \to N$ is Schatten p-harmonic where $p > \dim M$.

(1) Show that $Q(\mathrm{d}u)^{p-2}\,\mathrm{d}u \in W^{1,2}_{\mathrm{loc}}$. This holds if $M = N = \mathbf{H}^2$ [DU22, Theorem 4.17].
(2) Must $u \in W^{1,\infty}_{\mathrm{loc}}$?
(3) Must there exist $\alpha > 0$ such that $u \in C^{1+\alpha}_{\mathrm{loc}}$?

A map u is *Schatten ∞-harmonic* if it is a limit in C^0_{loc} of Schatten p-harmonic maps with the same boundary data. Schatten ∞-harmonic maps were introduced in [DU22], and their key feature is that $L := \mathrm{Lip}(u)$ is minimized among all maps with the boundary data f. The *canonical stretch locus* of u is the set of pairs $(x, y) \in M^2$ such that $x \neq y$ and for every Lipschitz map $v : M \to N$ with the same boundary data as u, if $\mathrm{Lip}(v) = L$, then $\mathrm{dist}(v(x), v(y)) = L\,\mathrm{dist}(x, y)$.

Problem 1.7 (Daskalopoulos–Uhlenbeck)**.** Let M, N be simply connected Riemannian manifolds such that N is complete and nonpositively curved. (To avoid technicalities, it may help to assume that M, N are both flat.) Let $u : M \to N$ be a Schatten ∞-harmonic map.

(1) Show that for every $(x, y) \in M^2$ such that $x \neq y$ and $\mathrm{dist}(u(x), u(y)) = \mathrm{Lip}(u)\,\mathrm{dist}(x, y)$, (x, y) is contained in the canonical stretch locus of u.
(2) Show that u is a *absolutely minimizing Lipschitz map* in the sense that for every convex $U \subseteq M$, and every Lipschitz map v which agrees with u away from U, $\mathrm{Lip}_U(u) \leq \mathrm{Lip}_U(v)$.
(3) Show that u is *tight* in the sense of Sheffield–Smart [SS12].

Problem 1.8 (Daskalopoulos–Uhlenbeck)**.** Let M be a simply connected Riemannian surface, let N a nonpositively curved Riemannian symmetric space, and let $\mathfrak{g}$ be the Lie algebra of Killing fields on N. Let $u_p : M \to N$ be Schatten p-harmonic maps with the same boundary data, converging to a Schatten ∞-harmonic map u. Since N is a symmetric space, one obtains a Noether current $\mathrm{d}v_q$ from u_p, where $1/p + 1/q = 1$ and v_q maps M into $\mathfrak{g}$.

Show that there exists $v : M \to \mathfrak{g}$ such that:

(1) Along a subsequence, $v_q \to v$ in $L^{3/2}_{\mathrm{loc}}$.
(2) $\mathrm{supp}\,|\mathrm{d}v|$ is a subset of the projection of the canonical stretch locus of u.

Thus the energy density of u_p concentrates on the canonical stretch locus of u. This holds whenever $M = N = \mathbf{H}^2$ and $\mathrm{d}u_p$ descends to a compact quotient of M [DU22, Theorem 7.1].

2. Harmonic analysis

For each p-form ψ on $\mathbf{R}^d$, the *exterior k-plane transform* of ψ is the function on the tautological bundle over the Grassmannian,

$$\mathcal{R}_k\psi(P, x) := \int_{P^\perp} \psi(x + y)|_P\,\mathrm{d}y$$

where $\psi(x + y)|_P$ is the projection of $\psi(x + y)$ to the pth exterior power of P.

Problem 2.1 (Solomon)**.** An inversion formula holds for the exterior k-plane transform on Schwartz p-forms [Sol11, Theorem 6.1]. Show that this inversion formula holds on the space of p-forms ψ, such that there exist $\varepsilon, \alpha > 0$ such that $\psi \in C^\alpha_{\mathrm{loc}}$ and $|\psi(x)| \lesssim \langle x\rangle^{-(d+\varepsilon)}$.

Given a compact set $X \subset \mathbf{R}^d$ and $h > 0$, let $X_h := \{x \in \mathbf{R}^d : \operatorname{dist}(x, X) < h\}$. Let

$$\mathscr{F}_h f(\xi) := (2\pi h)^{-d/2} \int_{\mathbf{R}^d} e^{-ix\cdot\xi/h} f(x)\,\mathrm{d}x$$

be the semiclassical Fourier transform. Let

$$\beta^\sharp(X, Y) := \sup\{\beta : \|1_{X_h} \mathscr{F}_h 1_{Y_h}\|_{L^2 \to L^2} \lesssim_\beta h^\beta\}.$$

denote the *sharp uncertainty exponent* of two compact sets $X, Y \subset \mathbf{R}^d$.

An *arithmetic Cantor set* is a compact set $X \subset \mathbf{R}$ such that $0 < \dim_{\mathcal{H}}(X) < 1$ and there exists an integer $M \geq 3$ and a set $A \subseteq \{0, \dots, M-1\}$ such that X is the set of $x \in \mathbf{R}$ such that there exists a sequence $(a_i) \subset A$ such that $x = \sum_{i=1}^\infty a_i/M^i$.

Problem 2.2 (Dyatlov)**.** Let $0 < \delta < 1$.

1. Show that there exists $\beta^\sharp_\delta > 0$ such that for every arithmetic Cantor set X, if $\dim_{\mathcal{H}}(X) = \delta$, then for the generic $\alpha > 0$,
$$\beta^\sharp(X, \alpha X) \geq \beta^\sharp_\delta.$$
2. Show that the above estimate fails for $\alpha = 1$.

An *Ahlfors-David set* is a compact set $X \subset \mathbf{R}^d$ such that, with $s := \dim_{\mathcal{H}}(X)$, for every $x \in X$ and $0 < r < 1$,

$$r^s \lesssim \mathcal{H}^s(X \cap B(x, r)) \lesssim r^s.$$

The sharp implied constant here is called the *Ahlfors-David regularity* of X.

Problem 2.3. Construct arithmetic Cantor sets X_j such that:

1. $\dim_{\mathcal{H}}(X_j) > 1 - 1/j$.
2. The Ahlfors-David regularity of X_j is bounded.
3. For some $\theta < 1$, $\beta^\sharp(X_j, X_j) \lesssim \theta^j$.

Problem 2.4 (Dyatlov)**.** Let $X \subset \mathbf{R}^2$ be an Ahlfors-David set such that $0 < \dim_{\mathcal{H}}(X) \leq 1$, and let $\chi \in C^\infty_{\mathrm{cpt}}((0, \infty))$. Show that there exists $\beta = \beta(X, \chi) > 0$ such that the following holds: Let

$$\mathcal{B}_h f(x) := \frac{1}{2\pi h} \int_{\mathbf{R}^2} |x - y|^{2i/h} \chi(|x - y|) f(y)\,\mathrm{d}y.$$

Then

$$\|1_{X_h} \mathcal{B}_h 1_{X_h}\|_{L^2 \to L^2} \lesssim h^\beta.$$

If this estimate fails, then X is *self-orthogonal* in the sense of [BLT23].

3. Differential geometry and geometric topology

Problem 3.1 (Liu)**.** Let M be a closed oriented Riemannian manifold of dimension d. Suppose that either $d \leq 7$ or the metric on M is suitably generic. Let $\rho \in H^{d-1}(M, \mathbf{R})$. If there exists a measurable $d-1$-form F such that $[F] = \rho$ and $\|F\|_{L^\infty} \leq 1$, must there exist a continuous $d-1$-form with these properties?

By a calibration argument, a positive answer to this question implies that every class in the image of the natural homomorphism $H_{d-1}(M, \mathbf{Z}) \to H_{d-1}(M, \mathbf{R})$ contains a smooth area-minimizing hypersurface. However, I expect that the proof would be utterly different than the usual proofs that minimal hypersurfaces are smooth.

Let $\mathscr{T}$ denote the Teichmüller space of a closed surface, let $\|\cdot\|_\infty$ denote the earthquake norm on the tangent bundle of $\mathscr{T}$, let ω be the Weil-Petersson symplectic form on $\mathscr{T}$, and let $\|\cdot\|_1$ denote the dual norm of $\|\cdot\|_\infty$ with respect to ω. By Wolpert's duality theorem, $\|\cdot\|_1$ is the infinitesimal version of the Thurston asymmetric metric on $\mathscr{T}$.

Problem 3.2. Let $\sigma \in \mathscr{T}$. Construct a map

$$\exp_\sigma : T_\sigma \mathscr{T} \to \mathscr{T}$$

with the following properties:

(1) On a neighborhood of 0, $\exp_\sigma$ is a diffeomorphism.
(2) For every ray ℓ based at 0, $(\exp_\sigma)_* \ell$ is a geodesic for the Thurston asymmetric metric.
(3) Let $v \in T_\sigma \mathscr{T}$ be such that $\|v\|_\infty = 1$, and let

$$v^* := \{\alpha \in T_\sigma \mathscr{T} : \omega(v, \alpha) = \|\alpha\|_1 = 1\}$$

be the dual flat of v. Then for every sufficiently small $t > 0$, v^* is the set of infinitesimal earthquakes generated by projective measured geodesic sublaminations of the canonical lamination maximally stretched by the homotopy class of

$$\mathrm{id}_M : (M, \sigma) \to (M, \exp_\sigma(tv)).$$

The "abelianized" version of this theorem (that is, for the stable norm) is true [Bac24b, §8.3].

4. Descriptive set theory and recursion theory

Problem 4.1. What is the algorithmic information density of the Gromov-Hausdorff space?

For our purposes, a sentence φ is *relatively consistent*, if the theory $\mathsf{ZFC}+\varphi$ is consistent provided that the theory ZFC + "There is a measurable cardinal" is consistent. The measurable cardinal is just to allow for the possibility that $2^{\aleph_0}$ is real-valued measurable; of course it would also be interesting to know that φ is consistent relative to ZFC alone.

Problem 4.2. A set $E \subseteq \mathbf{R}$ has a *small distance set* if

$$\dim(\{|x - y| : x, y \in E\}) = \dim E.$$

There exists $s_* < 1$ such that for every $\mathbf{\Sigma}^1_1$ set E such that $\dim_{\mathcal{H}} E \in [s_*, 1]$, E does not have a small distance set [Fal85]. On the other hand, if Martin's axiom is true, then for every $s \in [0, 1]$ there exists E_s such that E_s has a small distance set and $\dim_{\mathcal{H}} E_s = s$.

(1) Show that it is relatively consistent that there exists $s \in [0, 1]$ such that for every E, if $\dim E = s$ then E does not have a small distance set.
(2) What about $\mathbf{\Pi}^1_1$ and $\mathbf{\Sigma}^1_n$ sets?

Problem 4.3 (Fusco–Spector)**.** Let X be a Polish space, let $\mathcal{B}(X)$ be the Borel σ-algebra of X, and let $\mathcal{M}(X)$ be the space of finite signed Borel measures on X, equipped with its total variation norm. A function $\psi : \mathcal{B}(X) \to \mathbf{R}$ is an *integral representation* of a continuous linear functional L on $\mathcal{M}(X)$, if for every $\mu \in \mathcal{M}(X)$,

$$L(\mu) = \int_X \psi \, \mathrm{d}\mu,$$

where the integral is a Kolmogorov–Burkhill integral. A modification of the arguments of [Mau73] shows that assuming Martin's axiom, every continuous linear functional on $\mathcal{M}(X)$ has an integral representation.

Show that it is relatively consistent that there exists a continuous linear functional on $\mathcal{M}(X)$ which does not have an integral representation.

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