LINEAR PHASE AND WAVEFRONT SET FOR OSCILLATORY INTEGRALS

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1. Oscillatory integrals with linear phase

Recall that pseudodifferential operators are Fourier integral operators whose phase satisfies

$$\varphi(x, y, \theta) = \langle x - y, \theta\rangle.$$

A natural generalization is to consider Fourier integral operators for which φ is linear in θ. So, we need to discuss oscillatory integrals with linear phase.

Definition 1.1. A phase φ is a *linear phase* if there exists a smooth map $\Phi : X \to \mathbf{R}^N$ such that

$$\varphi(x) = \langle \Phi(x), \theta\rangle$$

and Φ has at least one zero.

The critical points of φ are exactly the zeroes of Φ. We want φ to have a critical point – Hörmander is working modulo C^∞, and integration with nonstationary phase implies that if φ has no critical points then any Fourier integral operator with phase φ is a smoothing operator, and so is 0 modulo C^∞.

Lemma 1.2. *Let $\varphi(x) = \langle \Phi(x), \theta\rangle$ be a linear phase on $X \subseteq \mathbf{R}^n$. Then Φ is a submersion in a neighborhood in $\{\Phi = 0\}$.*

Proof. Since φ is a phase and $\partial_\theta \varphi = \Phi$, if $\Phi(x) = 0$ then $d\Phi$ is surjective on tangent spaces. Thus the rank of Φ is N, but rank is lower semicontinuous so this remains true in a neighborhood of $\{\Phi = 0\}$. □

Since $\{\Phi = 0\}$ is nonempty, it follows that $N \leq n$ and the set $Y = \{\partial_\theta \varphi = 0\}$ is actually a manifold of codimension N, which we call the *critical manifold*.

Example 1.3. Suppose we are solving the wave equation forwards in time; then we get the phase $\varphi(x, t, \theta) = \langle x, \theta\rangle - t|\theta|$. Notice that this is singular along the light cone $\{(x, t) : |x| = t, t > 0\}$, which is a rectifiable set of codimension 1 but not a manifold. The takeaways here are that the case $N < n$ really is interesting in

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non-elliptic problems, and that linear phases really do generalize pseudodifferential operators in ways that general Fourier integral operators do not.

Our first result is that "up to isomorphism, a linear phase is determined by its critical manifold".

Lemma 1.4. *If φ_1, φ_2 are linear phases with the same critical manifold Y then there is a neighborhood U of Y and a map $\psi \in C^\infty(U \to GL(\mathbf{R}^N))$ such that for every $x \in U$,*

$$\varphi_1(x, \theta) = \varphi_2(x, \psi(x)\theta).$$

Proof. Since Φ_1, Φ_2 are submersions that cut out the same manifold, for every $x \in Y$, $d\Phi_j(x)$ have the same kernel and cokernel (and their cokernel is 0). Therefore there exists a linear automorphism $\psi(x)$ such that $\Phi_1(x) = \psi(x)\Phi_2(x)$ and so $\Phi_1 - \psi\Phi_2$ has double zeroes on Y. In particular ψ is smooth since Φ_j are, so we can extend ψ to a neighborhood of Y. If $\Phi_j = (\Phi_j^k)_k$ then Taylor's formula says that there exists a smooth family of matrices R

$$\Phi_1^j = \psi_{jk}\Phi_2^k + R_{jk}\Phi_2^k$$

where $R_{jk}|Y = 0$. Therefore $(\psi + R)^t$ has the required properties, at least when we are so close to Y that $||R|| < ||\psi||$ so that $\psi + R$ is invertible. $\square$

Conversely, if Y is a submanifold of X of codimension N, then we can write $Y = \{x \in X : x_1 = \cdots = x_N = 0\}$ in some coordinate system, and then use this fact to find a linear phase φ of critical manifold Y.

If φ_1, φ_2 are two linear phases with critical manifold Y, ψ is the isomorphism between them, and a_1 is a symbol, then

$$a_2(y, \theta) = a_1(y, \psi(y)\theta)|\det \psi(y)| \tag{1}$$

satisfies the equality of oscillatory integrals

$$\int_{\mathbf{R}^N} e^{i\varphi_1(y,\theta)} a_1(y, \theta)\ d\theta = \int_{\mathbf{R}^N} e^{i\varphi_2(y,\theta)} a_2(y, \theta)\ d\theta.$$

Most of this section will be dedicated to viewing symbols as a suitable map between *bundles* rather than something that obeys (1) under a transition map ψ.

Definition 1.5. Let Y be a submanifold of X, $N = \operatorname{codim} Y$, $1 - \rho \leq \delta < \rho$. Choose a linear phase φ of critical manifold Y. The space of all oscillatory integrals of the form

$$I(x) = (2\pi)^{-\frac{n+2N}{4}} \int_{\mathbf{R}^N} e^{i\varphi(x,\theta)} a(x, \theta)\ d\theta$$

where a is any element of $S_{\rho,\delta}^{m+(n-2N)/4}$ modulo $C^\infty(X)$ is called $I_{\rho,\delta}^m(X, Y)$.

Elements of $I^m_{\rho,\delta}(X,Y)$ are equivalence classes of distributions on X modulo $C^\infty(X)$, so $I^m_{\rho,\delta}(X,Y)$ is a subspace of $\mathcal{D}'(X)/C^\infty(X)$. The choice of φ does not matter, since such a φ exists, and applying the transformation ψ given by the previous lemma will at worst multiply a by a Jacobian which is constant in θ. Furthermore, integrating by nonstationary phase implies that we may assume that a is supported in a small neighborhood of Y. The strange choice of constant $-(n+2N)/4$ will be justified later on, but is partially motivated by this example:

Example 1.6. We're mainly interested in the case that I is the Schwartz kernel of a pseudodifferential operator on a manifold Y of dimension N. Then $X = Y^2$, so $n = 2N$. In this case, we can write $x = (y_1, y_2)$ and a is a function of $(y_1 - y_2, \theta)$ since the Schwartz kernel of a pseudodifferential operator is not just an oscillatory integral but a singular integral. Therefore we get the constant $(2\pi)^{-N}$ that appears in the definition of a pseudodifferential operator.

2. A review of differential topology

In order to talk about principal symbols, we will need to review some differential topology.

We first treat conormal bundles. Hörmander seems to mix up normal and conormal bundles a few times, which is pretty confusing.

Whenever I refer to a "closed embedding" I always mean a closed embedding of manifolds. Suppose that we have closed embedding $Y \subseteq X$. We want to define the normal bundle NY to Y to be the orthocomplement of TY in $TX|Y$, but this requires a choice of Riemannian metric. To get rid of this choice we instead observe that *if* we chose a Riemannian metric and ν was a normal vector, we could identify ν with a covector η using the Riesz representation theorem, and η would annihilate TY. So we can define the normal bundle to be the dual of the annihilator of TY. The annihilator of TY is a more "natural" concept, so we define:

Definition 2.1. If $Y \subseteq X$ is a closed embedding, the *conormal bundle* N^*Y to Y is the subbundle of T^*X defined by

$$N^*Y = \{(y,\eta) \in T^*X : y \in Y,\ \langle \eta, T_yY\rangle = 0\}.$$

Now we introduce the notion of a density bundle on a manifold X. We first do the same for a vector space. Suppose that V is a finite-dimensional vector space, and μ is the Haar measure of V. Then μ defines a linear map $\mu : V^{\otimes n} \to \mathbf{R}$ by letting $\mu(\bigotimes_j v_j)$ be the volume of the parallelpiped $\bigotimes_j v_j$, which satisfies for every linear operator A,

$$\mu(Av_1 \otimes \cdots \otimes Av_n) = |\det A| \mu(v_1 \otimes \cdots \otimes v_n).$$

Since Haar measures are unique only up to a scalar, we get a one-dimensional vector space of Haar measures, known as the density space of V.

Definition 2.2. Let X be a manifold of dimension n. The *s-density bundle*, Ω_s, on X is the line bundle associated to the $GL(\mathbf{R}^n)$-representation $A \mapsto |\det A|^{-s}$. An *s-density* is a smooth section of the s-density bundle.

More concretely, "a density is something that transforms like a density" in the sense that a is a s-density iff for every change of coordinates $y = \varphi(x)$,

$$a(y) = a(x)|\det d\varphi|^s.$$

In particular, a 1-density transforms like a volume form, and we can integrate 1-densities.

We obviously have $\Omega_s \otimes \Omega_t = \Omega_{s+t}$. Since we can integrate 1-densities, if u is a s-density and v is a $1-s$-density of compact support then $\langle u, v\rangle = \int_X u \otimes v$ is well-defined.

Definition 2.3. An *s-distribution density* on X is an element of the dual of $C_c^\infty(X \to \Omega_s)$.

3. Principal symbols for oscillatory integrals

We now assign principal symbols to the oscillatory integrals in $I^m_{\rho,\delta}(Y \times \mathbf{R}^N)$. I think that there is a typo in Hörmander's paper here where he mixes up n and N a few times.

Lemma 3.1. *The quantization map*

$$T : \frac{S^{m+\frac{n-2N}{4}}_{\rho,\delta}(Y \times \mathbf{R}^N)}{S^{m+\delta+\frac{n-2N}{4}-\rho}_{\rho,\delta}(Y \times \mathbf{R}^N)} \to \frac{I^m_{\rho,\delta}(X, Y)}{I^{m+\delta-\rho}_{\rho,\delta}(X, Y)}$$

that sends a symbol to its oscillatory integral with linear phase is a well-defined linear isomorphism.

Proof. Elements of $I^m_{\rho,\delta}(X, Y)/I^{m+\delta-\rho}_{\rho,\delta}$ are determined by the restriction of their symbol a to Y, so T to the oscillatory integral is a surjective and well-defined linear map. Let $Ta = 0$, and without loss of generality assume that:0

(1) a is supported in a neighborhood of Y.
(2) $Y = \{(0, y) \in X\}$ where we have the decomposition $x = (x', y)$.
(3) $\varphi(x, \theta) = \langle x', \theta\rangle$.

That $Ta = 0$ means that for every $u \in \mathcal{D}(X)$, $\langle Ta, u\rangle = 0$. But we can take $u(x)$ to only depend on x', thus $\langle Ta, u\rangle = 0$ implies

$$\iint_{Y \times \mathbf{R}^N} e^{i\langle x', \theta\rangle} a(x, \theta) u(x')\ dx'\ d\theta = 0.$$

Given $\xi \in \mathbf{R}^N$ we can replace $u(x')$ by $u(x')e^{-i\langle x',\xi\rangle}$ and conclude that

$$\iint_{Y\times\mathbf{R}^N} e^{i\langle x',\theta\rangle} a(x,\xi+\theta)u(x')\,dx'\,d\theta = 0. \tag{2}$$

We are interested in the asymptotics of (2) as $\xi \to \infty$. In fact, Taylor's formula gives

$$a(x,\xi+\theta) \sim \sum_\alpha \frac{i\partial_\xi^\alpha}{\alpha!} a(x,\xi)\theta^\alpha$$

and hence, taking the Fourier inversion of (2), we get

$$\begin{aligned}\iint_{Y\times\mathbf{R}^N} e^{i\langle x',\theta\rangle} a(x,\xi+\theta)u(x')\,dx'\,d\theta &\sim \sum_\alpha \frac{1}{\alpha!}\iint_{Y\times\mathbf{R}^N} e^{i\langle x',\theta\rangle} i\partial_\xi^\alpha a(x,\xi)\theta^\alpha u(x')\,dx'\,d\theta \\ &= (2\pi)^n \sum_\alpha \frac{(-\partial_{x'}^\alpha)}{\alpha!}(i\partial_\xi)^\alpha a(x,\xi)|_{x'=0} \\ &= (2\pi)^n a(0,y,\xi) \\ &\quad + (2\pi)^n \sum_{\alpha\neq 0} \frac{(-\partial_{x'}^\alpha)}{\alpha!}(i\partial_\xi)^\alpha a(x,\xi)|_{x'=0}.\end{aligned}$$

One has

$$\left((y,\xi) \mapsto (2\pi)^n \sum_{\alpha\neq 0} \frac{(-\partial_{x'}^\alpha)}{\alpha!}(i\partial_\xi)^\alpha a(x',y,\xi)|_{x'=0}\right) \in S_{\rho,\delta}^{m+\delta+\frac{n-2N}{4}-\rho}(Y\times\mathbf{R}^N)$$

which is 0 in the quotient space $S_{\rho,\delta}^{m+\frac{n-2N}{4}}(Y\times\mathbf{R}^N)/S_{\rho,\delta}^{m+\delta+\frac{n-2N}{4}-\rho}(Y\times\mathbf{R}^N)$ and hence so is a. □

It's tempting to define the principal symbol of an element of $I_{\rho,\delta}^m(X,Y)$ to be an element of $S_{\rho,\delta}^{m+\frac{n-2N}{4}}(Y\times\mathbf{R}^{\operatorname{codim} Y})/S_{\rho,\delta}^{m+\delta+\frac{n-2N}{4}-\rho}(Y\times\mathbf{R}^{\operatorname{codim} Y})$, but we need to deal with the transformation law (1) first.

Every linear phase φ induces a fiberwise isomorphism

$$\begin{aligned}\kappa_\varphi : Y\times\mathbf{R}^{\operatorname{codim} Y} &\to N^*Y \\ (x,\theta) &\mapsto d_x\varphi(x,\theta).\end{aligned}$$

Example 3.2. If the conormal bundle is nontrivial then κ_φ is clearly not an isomorphism of vector bundles. I think this is true for the Möbius band in $\mathbf{R}^3$ but I haven't checked it.

Let us identify $Y\times\mathbf{R}^{\operatorname{codim} Y}$ with N^*Y using κ_{φ_1}, so view symbols as functions on N^*Y. Suppose that we transform φ_1 to φ_2 by ψ, say $\psi(x)\theta_2 = \theta_1$. Then, if (a_1,φ_1) and (a_2,φ_2) define the same oscillatory integral, the transformation law (1) gives

$$a_2(x,\theta_2) = a_1(x,\psi(x)\theta_2)|\det\psi(x)| = a_1(x,\theta_1)|\det\psi(x)|.$$

Now we get rid of the Jacobian determinant. Fix $\varphi(x,\theta) = \langle \Phi(x), \theta\rangle$, and recall that Φ is a submersion near Y. Therefore the pullback distribution $\Phi^*\delta$, δ the Dirac distribution at 0 is well-defined. In local coordinates y, $\Phi^*\delta$ is just $|d\Phi(y)|^{-1}\prod_j dy_j$ (Theorem 6.1.3 in Hörmander's big book) which clearly transforms like a density on Y. Since $\mathbf{R}^{\operatorname{codim} Y}$ comes with the Lebesgue density $dV = \prod_j d\theta_j$, for every linear phase φ with critical manifold Y we get a density $\Phi^*\delta\ dV$ on $Y \times \mathbf{R}^{\operatorname{codim} Y}$. Then

$$D = (\kappa_\varphi)_*(\Phi^*\delta\ dV)$$

is a density on NY.

Let $\varphi_j(x,\theta) = \langle \Phi_j(x), \theta\rangle$ be linear phases with critical manifold Y which induce densities D_j. Suppose $\Phi_2 = \psi^t\Phi_1$. The transition map $\kappa(y,\theta) = (y, \psi^{-1}(y)\theta)$ satisfies $\kappa^* a_2 = |\det\psi| a_1$, but we also have $\kappa^* D_2 = |\det\psi|^{-2} D_1$. That is,

$$\kappa^* a_2\sqrt{D_2} = a_1\sqrt{D_1}.$$

Thus $a_1\sqrt{D_1}$ and $a_2\sqrt{D_2}$ are the same half-density, namely a bundle map

$$a\sqrt{D} \in S^{m+n/4}_{\rho,\delta}(N^*Y \to \Omega_{1/2}).$$

The fact that the symbol order is now independent of codimension is the other reason we defined $I^m_{\rho,\delta}(X,Y)$ so strangely.

Theorem 3.3. *Let $Y \subseteq X$ be a closed embedding of codimension N. Let $1-\rho \le \delta < \rho$, and choose a linear phase φ on $X \times \mathbf{R}^N$ with critical manifold Y. Let $I^m_{\rho,\delta}(X,Y)$ be the set of all distribution half-densities on X modulo $C^\infty(X)$ which are smooth on $X \setminus Y$ and are defined by oscillatory integrals*

$$Ta(x) = (2\pi)^{-\frac{n+2N}{4}} \int_{N^*_x Y} e^{i\varphi(x,\theta)} a(x,\theta)\ d\theta,$$

where a is a symbol of class $S^{m+(n-2N)/4}_{\rho,\delta}(N^*Y \to \Omega^{1/2})$ and the volume form $e^{i\varphi(x,\theta)}\ d\theta$ is defined by identifying the conormal space $N^*_x Y$ with $\mathbf{R}^N$ using the isomorphism $\theta \mapsto d_x\varphi(x,\theta)$. Then the quantization map

$$T : \frac{S^{m+\frac{n-2N}{4}}_{\rho,\delta}(N^*Y \to \Omega_{1/2})}{S^{m+\delta+\frac{n-2N}{4}-\rho}_{\rho,\delta}(N^*Y \to \Omega_{1/2})} \to \frac{I^m_{\rho,\delta}(X,Y)}{I^{m+\delta-\rho}_{\rho,\delta}(X,Y)}$$

is an isomorphism of vector spaces.

Proof. We just need to check that the above construction is invariant under changes of coordinates in X. This is just a consequence of the chain rule for half-densities used a bunch of times and not very interesting, so I'll omit it (it's page 119 in Hörmander's paper). □

Definition 3.4. With everything as in the previous theorem, we say that

$$a \in \frac{S^{m+\frac{n-2N}{4}}_{\rho,\delta}(N^*Y \to \Omega_{1/2})}{S^{m+\delta+\frac{n-2N}{4}-\rho}_{\rho,\delta}(N^*Y \to \Omega_{1/2})}$$

is the *principal symbol* of a distribution half-density A if $Ta = A$.

4. Wavefront sets

Some harmonic analysis review: sheet music tells us, for each time $t \in \mathbf{R}$, the amplitude (mezzoforte, pianissimo, etc.) that each frequency ($C\sharp$, F, etc.) should take at time t. This specifies a wave (i.e. a function) but in an overdetermined way, by the uncertainty principle. Still, it's often helpful to pretend as though a distribution u on $\mathbf{R}$ really is a function on $T^*\mathbf{R} \ni (t, \tau)$; namely, $u(t, \tau)$ denotes the amplitude of the note of pitch τ at time t. Wavefront sets are an example of this approach, where we are interested in both singularities in time and frequency.

Recall that the singular support of a distribution u only locates its singularities in time, and is defined by

$$\operatorname{sing\,supp} u = \bigcap_{\varphi u \in C^\infty} \{x \in X : \varphi(x) = 0\}$$

where φ ranges over cutoffs. That is, if φ cuts off u to a smooth function, then all the singularities of u must be in the closed set $\{\varphi = 0\}$. The idea is that "applying a pseudodifferential operator to a distribution is just like multiplying it in time-frequency space by the symbol", so we should be testing u against pseudodifferential operators that are cutoffs in time-frequency, rather than just cutoffs in time.

Now let A be a pseudodifferential operator of proper support and order 0 on X, and principal symbol a. Briefly we write $A \in L^0$. Recall that the characteristic set of A is

$$\gamma(A) = \{(x, \xi) : T^*X \setminus 0 : \liminf_{t \to \infty} |a(x, t\xi)| = 0\}.$$

Thus $\gamma(A)$ is a conic subset of T^*X, which "motivates" why we care about cone bundles. But it doesn't, really – why don't we just mod out by the $\mathbf{R}^+$ action on $T^*X \setminus 0$, since $\gamma(A)$ is clearly invariant under that action, and we only care about the directions of the covectors in $\gamma(A)$, rather than their magnitudes. Seriously, what is Hörmander doing here??

Definition 4.1. Let u be a distribution on X, and define the *wavefront set* of u to be

$$WF(u) = \bigcap_{\substack{Au \in C^\infty \\ A \in L^0}} \gamma(A).$$

Then $WF(u)$ is the intersection of closed conic sets and so is a closed conic set – but it's probably more helpful to think of as a closed subset of the cosphere bundle $(T^*X \setminus 0)/\mathbf{R}^+$.

Theorem 4.2. *Let $p : T^*X \to X$ be the natural projection. Then for every distribution u,*

$$p_*(WF(u)) = \operatorname{sing\,supp} u.$$

Moreover, if $x \in \operatorname{sing\,supp} u$, then the fiber $WF_x(u)$ of $WF(u)$ at x is the largest cone Γ in $T^*_x X$ such that for every cutoff φ to a neighborhood of x, there exists $N > 0$ such that for every $\xi \in \Gamma$,

$$|\widehat{\varphi u}(t\xi)| \gtrsim \langle t\xi \rangle^{-N}$$

as $t \to \infty$.

Here and always $\langle \xi \rangle = \sqrt{1 + \xi^2}$ is the Japanese angle bracket of ξ. I leave the proof for Ely to cover (or omit) next time. Let me just finish the talk with two examples.

Example 4.3. The term "wavefront set" derives from the following example. The Dirac measure δ on $\mathbf{R}^d$ has

$$WF(\delta) = T^*_0\mathbf{R}^d \setminus 0.$$

To see this, we first note that clearly $\operatorname{sing\,supp} \delta = \{0\}$. Taking the Fourier transform we get $\hat{\delta} = 1$, which doesn't decay in any direction, so every direction is included in the wavefront set.

If u is the solution of the wave equation with $u(0) = \delta$ and $u'(0) = 0$ then u is supported in the lightcone $\{(t, x) \in \mathbf{R}^{1+d} : x^2 = t^2\}$. (I think if $d = 0 \bmod 2$ then u is not literally the surface measure on the lightcone, because Huygens' principle is weak in this case.) Thus the lightcone is the wavefront of u. It is also the projection of $WF(u)$, which is the Hamiltonian flowout of $WF(\delta)$.

Example 4.4. Let U be an open subset of $\mathbf{R}^d$ such that ∂U is a smooth manifold, and let $u = 1_U$. Then $WF(u)$ is the conormal bundle of ∂U. Indeed, it is clear that $\operatorname{sing\,supp} u = \partial U$, and since ∂U is a smooth manifold, to compute $WF_x(u)$ we can flatten ∂U at x to assume that ∂U is a hyperplane $\{y = 0\}$, in which case $u(x, y) = H(y)$. If we consider Schwartz cutoffs $f(x, y) = g(x)h(y)$ then we get

$$\widehat{uf}(\xi, \eta) = \hat{g}(\xi)\widehat{Hh}(\eta)$$

If $\xi \neq 0$ then we clearly get decay in $\widehat{uf}(t\xi, t\eta)$ as $t \to \infty$ since $\hat{g}$ is a Schwartz function and

$$\widehat{Hh}(\eta) \lesssim \hat{H}(\eta) = \delta(\eta) + \frac{i}{\pi\eta} \sim 1/\eta$$

(in the sense of Cauchy principal value distributions). Meanwhile if $\xi = 0$ then $\widehat{uf}(t\xi, t\eta) = \hat{g}(0)\widehat{Hh}(t\eta) \sim 1/\eta$, so we get a singularity.