FUNCTIONS OF LEAST GRADIENT AND AREA-MINIMIZING LAMINATIONS

AIDAN BACKUS

1. Geodesic laminations and the ∞-Laplacian

I'll be saying a lot about PDE techniques for studying laminations, and the simplest kind of laminations are geodesic laminations, so let's start there.

Definition 1.1. A *geodesic lamination* in a Riemannian manifold M is a nonempty closed subset of M which has been partitioned into complete geodesics, called *leaves*.

A cross-section of a geodesic lamination could be homeomorphic to an arbitrary closed subset of $\mathbf{R}$ and we know that typically, this means it will look like a Cantor set.

Geodesic laminations typically show up whenever you are dealing with maps which minimize their Lipschitz constant. The Thurston school's approach to Teichmüller theory deals with minimizing Lipschitz maps between hyperbolic surfaces. In other applications of minimizing Lipschitz maps, such as to optimal transport, or to Farre–Landesberg–Minsky's analysis of horocycle orbit closures, we want to look at scalar-valued minimizing Lipschitz functions, $u : M \to \mathbf{R}$. In that case, $\mathrm{Lip}(u) = \|\,\mathrm{d}u\|_{L^\infty}$ and this suggests that we can find a minimizing Lipschitz map by minimizing $\|\,\mathrm{d}u\|_{L^p}$ and taking $p \to \infty$. In fact, if u minimizes $\|\,\mathrm{d}u\|_{L^p}$ it solves the *p-Laplacian*,

$$0 = \mathrm{d}^*(|\,\mathrm{d}u|^{p-2}\,\mathrm{d}u) = (p-2)|\,\mathrm{d}u|^{p-4}\langle \nabla^2 u, \mathrm{d}u \otimes \mathrm{d}u\rangle + |\,\mathrm{d}u|^{p-2}\Delta u.$$

Renormalizing and taking $p \to \infty$ we get the *∞-Laplacian*,

$$0 = \langle \nabla^2 u, \mathrm{d}u \otimes \mathrm{d}u\rangle$$

and the (viscosity) solutions of this PDE minimize their Lipschitz constant. Using viscosity solutions techniques, one can show:

Theorem 1.2 (Crandall and Daskalopoulos–Uhlenbeck)**.** *Let u be ∞-harmonic. If $|\,\mathrm{d}u|$ attains its maximum, then it does so on a geodesic lamination λ, and for every leaf $\gamma \subseteq \lambda$, $u \restriction \gamma$ is an affine function with slope* $\mathrm{Lip}(u)$*.*

Corollary 1.3. *Let M be closed, and let ρ be a homotopy class of maps $M \to \mathbf{S}^1$. Then the immersed complete curves γ such that for every minimizing Lipschitz map f in ρ, $f \restriction \gamma$ is an affine function with slope* $\mathrm{Lip}(f)$*, form a geodesic lamination.*

λ is called the *canonical lamination* maximally stretched by ρ. To see why if $u \restriction \gamma$ is an affine function with slope $\mathrm{Lip}(u)$, that curve should be a geodesic, suppose that γ has endpoints x, y, then

$$\mathrm{len}(\gamma) = \frac{1}{\mathrm{Lip}(u)} \int_\gamma \mathrm{d}u = \frac{1}{\mathrm{Lip}(u)} \int_{[x,y]} \mathrm{d}u = \mathrm{dist}(x, y).$$

Specializing to maps from a closed surface M to $\mathbf{S}^1$, let

$$\mathrm{d}v_q := |\,\mathrm{d}u_p|^{p-2} \star \mathrm{d}u_p$$

where $1/p + 1/q = 1$. Thus $\mathrm{d}v_q$ is closed, so we get a function v_q. Moreover

$$\mathrm{d}^*(|\,\mathrm{d}v_q|^{q-2}\,\mathrm{d}v_q) = \mathrm{d}^*(|\,\mathrm{d}u_p|^{(p-1)(q-2)+(p-2)} \star \mathrm{d}u_p) = \mathrm{d}^2 u_p = 0$$

so v_q is q-harmonic. As $q \to 1$, $v_q \to v$ in $L^{3/2}_{\mathrm{loc}}$, where the level sets $\partial\{v > t\}$ of v are all leaves of the canonical lamination λ. In other words, $|\,\mathrm{d}v|$ is a *transverse measure* to λ.

Thurston and Gueritaud–Kassel have constructed a canonical maximally stretched lamination for certain homotopy classes of maps between closed hyperbolic surfaces, but the proof is completely different, based on triangle comparison arguments. The analogue of the p-Laplacian for approximating minimizing Lipschitz maps between manifolds is the *Schatten p-Laplacian*,

$$\mathrm{D}_u^*((\mathrm{d}u\,\mathrm{d}u^\dagger)^{\frac{p-2}{2}}\,\mathrm{d}u) = 0.$$

We can't generalize Theorem 1.2 because the Schatten p-Laplacian is a system of PDE, so viscosity solutions don't make sense. Daskalopoulos–Uhlenbeck have shown, however, that we can use the above argument to construct a transverse measure on Thurston's canonical lamination and this has a few applications in Teichmüller theory. However, this line of research seems really hard to make progress on, because the Schatten p-Laplacian is arguably beyond the current methodology for studying elliptic PDE.

2. Functions of least gradient

Going back to maps from a surface to $\mathbf{S}^1$ for the time being, let's more carefully scrutinize more carefully the transverse measure $|\,\mathrm{d}v|$. We got v as a limit of q-harmonic functions v_q and those all minimize $\int |\,\mathrm{d}v_q|^q\,\mathrm{d}V$. So v minimizes $\int |\,\mathrm{d}v|\,\mathrm{d}V$.

What do I mean by $|\,\mathrm{d}v|$? $W^{1,1}$ does not admit a weakstar topology, so we don't expect $v \in W^{1,1}$ (in fact, if M is a closed, negatively curved, surface, then $v \notin W^{1,1}$). So $|\,\mathrm{d}v|$ is just a Radon measure, not a function. The space of functions with $\int |\,\mathrm{d}v|\,\mathrm{d}V < \infty$ is called BV for *bounded variation*.

Definition 2.1 (de Giorgi and Miranda)**.** A function $v \in BV$ has *least gradient* if u minimizes $\int |\,\mathrm{d}v|\,\mathrm{d}V$ subject to a boundary condition.

Let's look at least gradient functions on any Riemannian manifold M of dimension $d \leq 7$, not just a surface. The most important property they have is that, because of the coarea formula,

$$\int_M |\,\mathrm{d}v|\,\mathrm{d}V = \int_{-\infty}^{\infty} \mathrm{Area}(\partial\{v > t\})\,\mathrm{d}t,$$

the level sets $\partial\{v > t\}$ are area-minimizing. Because of this property, functions of least gradient got a lot of attention from the Italian GMT school in the 1960s. This culminated in Bombieri–de Giorgi–Giusti's refutation of the Bernstein conjecture.

Because BV is nonreflexive, solvability of the Dirichlet problem for a function of least gradient, even in the simplest case that the domain is a disk in $\mathbf{R}^2$, is quite subtle. Only over the past decade (Jerrard–Moradifam–Nachman, Górny, B) have we obtained satisfactory results. These proofs rely on the local GMT of minimal hypersurfaces (eg, the maximum principle for area-minimizing currents).

Definition 2.2. A *lamination* λ in M of codimension 1 is a nonempty closed set $\operatorname{supp}\lambda \subseteq M$, which is partitioned into complete hypersurfaces called *leaves*, and a maximal atlas of Lipschitz charts in which $\operatorname{supp}\lambda$ takes the form $K \times (0,1)^{d-1}$ where K is a closed subset of $\mathbf{R}$, such that every leaf takes the form $\{k\} \times (0,1)^{d-1}$ where $k \in K$. The lamination λ is *minimal* if every leaf of λ has mean curvature 0.

So we imposed a Lipschitz regularity condition. One can show that every geodesic lamination in a surface satisfies this condition but in general you have to separately impose it.

Theorem 2.3 (B)**.** *Let $\mathcal{S}$ be a nonempty set of disjoint injectively immersed minimal hypersurfaces of locally uniformly bounded $\mathrm{I\!I}$ and closed union. Then $\mathcal{S}$ is the set of leaves of a lamination.*

Solomon proved this for foliations, using the Harnack inequality, and Colding–Minicozzi sketched the idea of how to extend the argument to laminations. The $\mathrm{I\!I}$ requirement is caused by subtleties of varifold convergence (and it might be possible to remove it with more work), but in our application it will always be easy to check.

Theorem 2.4 (B)**.** *Let $v \in BV_{\mathrm{loc}}(M)$ be nonconstant. Then v locally has least gradient iff the level sets of v form a minimal lamination λ whose leaves have locally uniformly bounded $\mathrm{I\!I}$.*

In particular, $|\,\mathrm{d}v|$ is a transverse measure to λ.

Most of the work of the proof is dealing with difficulties of BV functions, but conceptually the plan of the proof of Theorem 2.4 is pretty simple. If v is assumed to have least gradient, we want to check the hypotheses of Theorem 2.3. Since the level sets of v are minimal hypersurfaces, we can use a maximum principle argument to ensure that they are all disjoint. Also, by assumption we can find an open cover in which every level set of v is area-minimizing, hence stable and of bounded area; we then use Schoen–Simon's estimates on stable minimal hypersurfaces to get the required bound on $\mathrm{I\!I}$. Conversely, if the level sets of v are assumed to be a minimal lamination, we can use the bound on $\mathrm{I\!I}$ to find an open cover by sets in which the level sets are all area-minimizing, and then use the coarea formula.

Problem 2.5. Suppose that v is undergoing total variation flow. Under what conditions on the boundary and initial data do the level sets of v form a laminations of hypersurfaces, each undergoing mean curvature flow?

3. The canonical calibrated lamination

Suppose that M is closed and oriented.

Definition 3.1 (Harvey–Lawson)**.** A *calibration* of codimension 1 is a closed $d-1$-form of L^∞ norm 1. If F is a calibration, a hypersurface N is *F-calibrated* if F pulls back to the area form on N.

A calibration is basically a "certificate" that a hypersurface is area-minimizing. Indeed, if a closed hypersurface N is F-calibrated and $N' \sim N$,

$$\mathrm{Area}(N) = \int_N F = \int_{N'} F \le \mathrm{Area}(N').$$

This is just the same argument as in the Lipschitz maps case. If you go on Pawn Stars to try to sell a minimal hypersurface N and you don't have a calibration for N, Rick Harrison will call up one of his buddies who will say that it's not area-minimizing. "This just looks like a critical point of the area functional, best I can do is 50 bucks." But if you try to sell Rick N and *and a calibration for N*, he'll be very impressed.

We are going to use Theorem 2.4 to get a generalization of the canonical maximally stretched lamination, and in this theorem we set for every $\rho \in H^{d-1}(M, \mathbf{R})$,

$$\|\rho\|_\infty := \inf_{[F]=\rho} \|F\|_{L^\infty}.$$

Theorem 3.2 (B)**.** *Let $\rho \in H^{d-1}(M, \mathbf{R})$ and $\|\rho\|_\infty = 1$. Then the immersed complete hypersurfaces N such that for every calibration F in ρ, N is F-calibrated, form a minimal lamination λ_ρ.*

λ_ρ is the *canonical lamination* calibrated by ρ. Let's look at the proof when $d = 2$ for simplicity. Let F be a calibration, so $F = \mathrm{d}u$ for some Lipschitz function u on $\tilde{M}$ such that $\mathrm{Lip}(u) = 1$. Since $\|\rho\|_\infty = 1$, u has minimizing Lipschitz constant, so we're actually giving a new proof of Theorem 1.2, but the proof we're about to give is pretty different (since no viscosity solutions are available in general).

ρ might not contain any continuous calibrations (at least I don't know that!) so we have to use continuity of u as a proxy for continuity of F. This is a key difficulty of the proof, and in general it comes down to Anzellotti's theory of compensated compactness for BV, but let's pretend that F is continuous.

Let γ be an immersed complete curve such that $u \restriction \gamma$ is affine of slope 1. Thus γ is F-calibrated and so is a geodesic. If another such curve intersects γ at a point x, then they have the same tangent vector $F^\sharp(x)$, so by the maximum principle they are equal. Therefore we can use Theorem 2.3 to show that the set of all such curves is a geodesic lamination λ_F, *if such a curve exists*. Putting this aside for now...

To build λ_ρ we take the intersection of the sets of leaves of the λ_F where F ranges over all calibrations. We have to show this intersection is nonempty and to do this we use the *stable norm* on $H_{d-1}(M, \mathbf{R})$,

$$\|\alpha\|_1 := \inf_{[N]=\alpha} \operatorname{Area}(N).$$

This is the dual norm of $\|\cdot\|_\infty$, and it doesn't have to be strictly convex. We look at the dual flat

$$\rho^* := \{\alpha \in H_{d-1}(M, \mathbf{R}) : \langle \rho, \alpha \rangle = \|\alpha\|_1 = 1\}.$$

Given $\alpha \in \rho^*$, we can find a function v of least gradient which is "suitably twisted by α" in the sense that for any F,

$$\int_M F \wedge \mathrm{d}v = \langle \rho, \alpha \rangle = \|\alpha\|_1 = \int_M |\,\mathrm{d}v|\,\mathrm{d}V,$$

and apply Theorem 2.4 to obtain a lamination whose leaves are all then F-calibrated for every F.

Theorem 3.3 (B). *ρ^* is the set of homology classes of projective measured sublaminations of λ_ρ.*

This falls out from the previous proof. The ergodic theory of laminations then implies that ρ^* is a convex polytope whose vertices correspond to ergodic sublaminations of λ_ρ. Furthermore, a vertex α has rational direction iff it corresponds to a closed hypersurface.

Corollary 3.4. *If the stable norm ball is strictly convex, then there exists a uniquely ergodic minimal lamination without a closed leaf.*

Proof. Let $\alpha \in H_{d-1}(M, \mathbf{R})$ have $\|\alpha\|_1 = 1$ and irrational direction, and choose ρ so that $\rho^* = \{\alpha\}$. Then λ_ρ only admits one measured sublamination κ, and κ is as desired. □

This suggests that it would be nice to have topological criteria for the stable norm ball to be strictly convex, and the canonical lamination gives us a few of these, and here is one, which was proposed by Auer–Bangert in a research announcement 25 years ago.

Corollary 3.5. *If $[\alpha, \beta] \subset H_{d-1}(M, \mathbf{R})$ is a line segment in the stable unit sphere, then the intersection product of α, β is 0.*

Proof. There exists $\rho \in H^{d-1}(M, \mathbf{R})$ such that $[\alpha, \beta] \subseteq \rho^*$. So there are measured sublaminations $\kappa_\alpha, \kappa_\beta \subseteq \lambda_\rho$ which represent α, β. So $\kappa_\alpha, \kappa_\beta$ do not intersect, except if they have common leaves. □

4. The earthquake norm

Here's a definition of the earthquake norm that doesn't mention earthquakes. Let M be a closed hyperbolic surface, and let $\mathscr{F}$ be the sheaf of Killing fields on M. Then the sheaf cohomology, $H^1(M, \mathscr{F})$, is canonically isomorphic to the set of measured geodesic laminations in M. The *earthquake norm* of a cohomology class is the mass of the corresponding lamination, and the *intersection number* of a pair of classes is the intersection number of the corresponding laminations.

After proving Corollary 3.5, I looked at the arXiv, and to my surprise, the following theorem had been posted a few weeks prior.

Theorem 4.1 (Huang–Ohshika–Pan–Papadopoulos)**.** *If* $[\alpha, \beta] \subset H^1(M, \mathscr{F})$ *is a line segment in the earthquake unit sphere, then the intersection number of* α, β *is* 0*.*

In fact, there are a number of other theorems suggested by Auer–Bangert which can be proven using the canonical calibrated lamination. Each of these has an analogue proven by Thurston's school for the earthquake norm proven using the canonical maximally stretched lamination. The remarkable thing is that these theories seem to have been developed totally independently. There's no hint of the Thurston school's work or the canonical lamination in Auer–Bangert's work, and conversely much of the Thurston school's work predates Auer–Bangert. For example:

Theorem 4.2 (Thurston school)**.** *Every maximal flat of the earthquake unit sphere is a polytope.*

The point of the proof is to think of a maximal flat as the set of homology classes of measured sublaminations of some lamination λ with mass 1. In the case of the stable norm, λ is canonical, but I don't know that for the earthquake norm, and one issue here is that we don't have a completely satisfactory answer to:

Problem 4.3 (Pan–Wolf)**.** What is the correct notion of exponential map for the Thurston asymmetric metric on Teichmüller space?