

# NOTES ON THE FRACTAL UNCERTAINTY PRINCIPLE

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## CONTENTS

1. Motivation from spectral theory	1
1.1. Gamow’s resonant states	1
1.2. The Dyatlov–Zahl theorem	2
2. The fractal uncertainty principle	3
2.1. Statement of the theorem	3
2.2. Some examples	4
3. Model problem: Arithmetic Cantor sets	4
4. The Dyatlov–Jin uncertainty principle	6
4.1. Discretization of compactly supported doubling measures	7
4.2. Getting a gain at one scale	9
4.3. Induction	12
References	13

## 1. MOTIVATION FROM SPECTRAL THEORY

1.1. **Gamow’s resonant states.** Consider an  $\alpha$ -particle – that is, a He-4 nucleus that was just ejected from the nucleus of a heavy atom (eg, of U-238). The physicist Gamow thought of the  $\alpha$ -particle as a wavefunction  $u$ , it solves a Schrödinger equation

$$(i\partial_t + P)u = 0 \tag{1.1}$$

on all of  $\mathbf{R}^d$ , where  $P$  is uniformly elliptic on  $\{|x| \gtrsim 1\}$  (and in fact converges to a rescaled Laplacian near infinity) but may degenerate near the origin due to effects of the strong and electromagnetic forces near the U-238 nucleus.

According to quantum mechanics, steady states of (1.1) are eigenfunctions of  $P$  with  $L^2$  norm one. But  $P$  is basically just the Laplacian, so it has essential spectrum, and so (1.1) has no steady states. Gamow got around this conundrum by considering functions which are formally eigenfunctions of  $P$ , as detected by the resolvent of  $P$ . Recall that the resolvent of an operator  $P$  is given by

$$R_P(z) = (P - z^2)^{-1}$$

and, using the uniform ellipticity of  $P$ , it admits an analytic continuation to  $\mathbf{C}$  minus a locally finite set of poles [DZ19].

**Definition 1.1.** A *resonance* of  $P$  is a pole of  $R_P$  (counted with multiplicity).

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Resonances  $z$  corresponds to particles, called resonant states, which are sort of like formal eigenfunctions of  $P$  with eigenvalue  $z^2$ . Such a particle has energy  $\operatorname{Re} z$  and half-life  $-h/\operatorname{Im} z$ . (Here  $h := \hbar/\sqrt{2m}$  where  $m$  denotes mass.)

In general  $P$  may admit lots of resonant states, but if  $-\operatorname{Im} z \gg h$ , they are not very useful, as they are particles of very short half-life, and there are infinitely many distinct such particles. We want a condition which guarantees that there's only finitely many resonant states worth caring about.

**Definition 1.2.**  $P$  has an *essential spectral gap* of size  $\beta \geq 0$  if there are only finitely many resonances with  $\operatorname{Im} z \geq -\beta$ .

**1.2. The Dyatlov–Zahl theorem.** Taking  $h \rightarrow 0$  is called the *semiclassical limit* and corresponds to approximating the quantum world by its classical counterparts. In that case particles are expected to move along geodesics, if  $P$  is the Laplace-Beltrami operator of some Riemannian metric. So the study of resonant states would be especially interesting if we assumed that  $P$  was the Laplace-Beltrami operator of a negatively curved manifold, since then the geodesic flow would be chaotic: two particles could start with very similar positions and momenta, and end up in very different places in an exponentially small timeframe.

Let  $\Gamma$  be a “nice” discrete subgroup of the properly orthochronous Lorentz group  $SO^+(1, d+1)$ . This is the group of all spacetime symmetries which preserve orientation and the arrow of time. In particular it acts on the future unit hyperboloid  $\{t = \sqrt{|x|^2 + 1}\}$  by isometries. But that hyperboloid is hyperbolic space  $\mathbf{H}^{d+1}$ . So we can take the manifold

$$M := \mathbf{H}^{d+1}/\Gamma$$

whose fundamental group is then  $\Gamma$ .

Let's assume that  $M$  has infinite ends. It's negatively curved, so its geodesic flow is chaotic, but also some geodesics are trapped, while some escape to infinity. Thus we have a highly unpredictable geodesic flow, and the geodesics which do escape correspond to trajectories of resonant states. So, if we understand the group  $\Gamma$ , we should be able to say some interesting things about the resonant states.

Dyatlov–Zahl made the above heuristics rigorous using Hörmander's theory of Fourier integral operators and singularity propagation [DZ16].

**Theorem 1.3** (Dyatlov–Zahl '16). *Let  $\Gamma$  be a convex cocompact discrete subgroup of  $SO^+(1, d+1)$ . Let  $\Lambda_\Gamma$  be the boundary of an orbit of  $\Gamma$  in the sphere at infinity  $\mathbf{S}^d := \partial\mathbf{H}^{d+1}$ . Let*

$$B_{\chi,h}f(y) := h^{d/2} \int_{\mathbf{S}^d} |y-x|^{2i/h} \chi(x,y) f(x) dx$$

where  $\chi$  has compact support in the off-diagonal of  $(\mathbf{S}^d)^2$ . If

$$\|1_{\Lambda_\Gamma+B_h} B_{\chi,h} 1_{\Lambda_\Gamma+B_h}\|_{L^2(\mathbf{S}^d) \rightarrow L^2(\mathbf{S}^d)} \lesssim_\chi h^\beta,$$

then the Laplacian on  $\mathbf{H}^{d+1}/\Gamma$  has an essential spectral gap  $\geq \beta$ .

Notice that  $B_{\chi,h}$  formally resembles the Fourier transform, since

$$|y-x|^{2i/h} = e^{i\Phi(x,y)/h}$$

where  $\Phi(x,y) = \log|x-y|$  is a sort of “generalized inner product” which degenerates away from the support of  $\chi$ . We will focus on this model case in the sequel. Also notice that  $\Lambda_\Gamma$  is a fractal (topologists love these sets).

2. THE FRACTAL UNCERTAINTY PRINCIPLE

2.1. **Statement of the theorem.** What does it mean for a set to be fractalline? What do we mean when we say that  $\Lambda_\Gamma$  is a fractal? Well, we want some sort of self-similarity, and self-similarity at all scales is what the Ahlfors-David condition says:

**Definition 2.1.** A compact set  $X \subset \mathbf{R}^d$  of Hausdorff dimension  $\delta$  is *Ahlfors-David regular* if its Hausdorff measure  $\mu$  satisfies

$$\mu(B(x, r)) \sim r^\delta.$$

If  $\Gamma$  is as in the Dyatlov–Zahl theorem, then  $\Lambda_\Gamma$  is  $\delta$ -regular for some  $\delta \in (0, \delta)$ .

**Theorem 2.2** (fractal uncertainty principle). *Let  $X$  be  $\delta$ -regular,  $X_h := X + B_h$ , and*

$$\mathcal{F}_h f(y) := h^{d/2} \int_{\mathbf{R}^d} e^{-ix \cdot y/h} f(x) dx.$$

*Assuming ???, there exists  $\beta \geq \beta_0$  such that*

$$\|1_{X_h} \mathcal{F}_h 1_{X_h}\|_{L^2(\mathbf{R}^d) \rightarrow L^2(\mathbf{R}^d)} \lesssim h^\beta.$$

Here’s how to interpret this theorem. Suppose that  $f$  rapidly decays away from a small neighborhood  $X_h$  of  $X$ . Then  $1_{X_h}$  is basically just  $f$ , and after we take the Fourier transform  $\mathcal{F}_h f \approx \mathcal{F}_h 1_{X_h} f$ , the definition of the fractal uncertainty principle indicates that what’s left must avoid  $X_h$ , in the sense that its contraction with  $1_{X_h}$  is tiny in  $L^2$ .

OK so we need to say what ??? and  $\beta_0$  are. First observe that

$$\|1_{X_h} \mathcal{F}_h 1_{X_h}\| \leq \|1_{X_h}\|_{L^\infty}^2 \|\mathcal{F}_h\|_{L^2 \rightarrow L^2} \leq 1 = h^0$$

by Plancherel’s theorem and Hölder’s inequality. So if FUP is going to be an interesting theorem, we must have  $\beta_0 \geq 0$ .

We have  $X_h \approx \bigcup_{n=1}^N B(x_n, h)$  where  $\{x_n\}$  is a maximal  $O(h)$ -separated subset of  $X$ . So  $|X_h| \sim Nh^d$ , but  $\mu(B(x_n, h)) \sim h^\delta$  and  $\mu(X) \sim 1$ , hence  $N \sim h^{-\delta}$  and so

$$|X_h| \sim h^{d-\delta}.$$

In particular we have

$$\|1_{X_h}\|_{L^2} \lesssim h^{\frac{d-\delta}{2}},$$

we so

$$\begin{aligned} \|1_{X_h} \mathcal{F}_h 1_{X_h}\| &\leq \|1_{X_h}\|_{L^\infty \rightarrow L^2} \|\mathcal{F}_h\|_{L^1 \rightarrow L^\infty} \|1_{X_h}\|_{L^2 \rightarrow L^1}, \\ &\leq \|1_{X_h}\|_{L^2}^2 \|\mathcal{F}_h\|_{L^1 \rightarrow L^\infty} \\ &\lesssim h^{d-\delta} h^{-\frac{d}{2}} = h^{\frac{d}{2}-\delta}. \end{aligned}$$

Thus we have

$$\beta_0 := \max\left(0, \frac{d}{2} - \delta\right),$$

the so-called *Patterson-Sullivan exponent*.

Now, what is ???:

- Dyatlov–Zahl ’16 [DZ16]:  $d = 1, \delta = \frac{1}{2}$ .
- Bourgain–Dyatlov ’18 [BD18]:  $d = 1, \frac{1}{2} \leq \delta < 1$ .
- Dyatlov–Jin ’18 [DJ18]:  $d = 1, 0 < \delta \leq \frac{1}{2}$ .

- Han–Schlag '20 [HS20]:  $\frac{d}{2} \leq \delta < d$ ,  $X = \prod_n X_n$  where  $X_n \subset \mathbf{R}$  is  $\delta_n$ -regular,  $\delta_n \in (0, 1)$ .
- Cladek–T. Tao '21 [CT21]:  $d$  odd,  $\delta = \frac{d}{2}$ .
- Cohen '22 [Coh22]:  $d = 2$ ,  $1 \leq \delta < 2$ ,  $X$  is an arithmetic Cantor set (to be defined later) which does not contain a line.
- Backus–Leng–Z. Tao '23:  $0 < \delta \leq \frac{d}{2}$  and  $X$  is not orthogonal to itself (to be defined later).

Roughly speaking there are three cases:

- The case  $0 < \delta \leq \frac{d}{2}$  is proven using the self-similarity to get lots of cancellations in the Fourier phase; this is called *Dolgopyat's method*. We'll prove it in detail later.
- The case  $\delta = \frac{d}{2}$  is proven using additive combinatorics.
- The case  $\frac{d}{2} \leq \delta < d$  is proven using complex analytic or algebro-geometric methods. This case is hardest IMO.

**2.2. Some examples.** The above theorem is not quite sharp but there are some improvements that we cannot make to it.

**Example 2.3.** Let  $X = [-3, 3]$ ,  $d = 1$ . Then  $e^{-x^2/2}$  is microlocalized to  $X$  at scale  $h$  (if  $h$  is small enough), so we cannot take  $\delta = d$ .

**Example 2.4.**  $X = \{0\}$ . Then  $e^{-|x|^2/(2h)}$  is microlocalized to  $X$  at scale  $h$ , so we cannot take  $\delta = 0$ .

**Example 2.5.** Let  $X$  be the union of two orthogonal line segments,  $d = 2$ . Then  $e^{-x^2/2-y^2/(2h)}$  is microlocalized to the horizontal line segment and its Fourier transform is microlocalized to the vertical part. So  $X$  cannot have too much “linearly independent additive structure”, whatever on god's green earth that means.

### 3. MODEL PROBLEM: ARITHMETIC CANTOR SETS

A lot of the ideas of the proof already appear in the case of so-called arithmetic Cantor sets, which are so self-similar that the technicalities simplify a lot.

**Definition 3.1.** Let  $L \geq 3$  be an integer and  $A \subset \{1, \dots, L\}^d$  be a nonempty proper subset, which we think of as labelling the  $L$ -adic subcubes of a given cube. The *arithmetic Cantor set* generated by  $A$  is defined by repeatedly splitting a cube into  $L$ -adic subcubes, and keeping those closed subcubes which belong to  $A$ .

**Example 3.2.** The middle-thirds Cantor set is arithmetic, with  $L = 3$  and  $A = \{1, 3\}$ . If  $\theta \in (0, 1)$  is irrational, the middle- $\theta$  Cantor set is not arithmetic.

At scale  $N^{-1} := L^{-k}$ , an arithmetic Cantor set looks like  $N^{-1}C_{k,A}$  where

$$C_{k,A} := \{a_0 + a_1L + \dots + a_kL^k : a_i \in A\}.$$

In particular it has dimension  $\delta := \frac{\log \text{card } A}{d \log L}$ . Thus, the arithmetic Cantor set satisfies the fractal uncertainty principle if there exists  $\varepsilon_0 > 0$  such that the discrete Fourier transform

$$\mathcal{F}_N f(\xi) := N^{-d/2} \sum_{x \in \{1, \dots, N\}^d} \exp\left(2\pi i \frac{x \cdot \xi}{N}\right) f(x)$$

satisfies

$$\|1_{C_{k,A}} \mathcal{F}_N 1_{C_{k,A}}\|_{\ell^2 \rightarrow \ell^2} \lesssim N^{-\beta - \varepsilon_0} \quad (3.1)$$

where  $\beta := \max(0, \frac{d}{2} - \delta)$  is the Patterson-Sullivan exponent.

**Proposition 3.3.** *Suppose that  $\delta \in (0, d/2]$  and there exist  $x, x', \xi, \xi' \in A$  such that  $\langle x - x', \xi - \xi' \rangle \neq 0$  modulo  $N$ . Then there exists  $\varepsilon_0 > 0$  such that the arithmetic Cantor set satisfies the fractal uncertainty principle (3.1).*

*Proof.* Since  $\delta \leq d/2$ , if we can beat the exponent  $\frac{d}{2} - \delta$ , then we automatically beat the Patterson-Sullivan exponent  $\beta$ .

We now observe the following *induction on scale* inequality:

$$\|1_{C_{k_1+k_2,A}} \mathcal{F}_{N_1 N_2} 1_{C_{k_1+k_2,A}}\| \leq \|1_{C_{k_1,A}} \mathcal{F}_{N_1} 1_{C_{k_1,A}}\| \|1_{C_{k_2,A}} \mathcal{F}_{N_2} 1_{C_{k_2,A}}\|. \quad (3.2)$$

It follows from the fact that if we have  $u, v \in \ell^2([N]^d)$  and  $\xi_1, x_2 \in [N]^d$ , we can define  $u_{\xi_1} \in \ell^2([N_2]^d)$  and  $v_{x_2} \in \ell^2([N_1]^d)$  by  $u_{\xi_1}(\xi_2) := u(\xi_1 + N_1 \xi_2)$  and  $v_{x_2}(x_1) := v(x_2 + N_2 x_1)$ , and then we have the *decimation* formula

$$\langle \mathcal{F}_N u, v \rangle = \sum_{\xi_1 \in [N_1]^d} \sum_{x_2 \in [N_2]^d} \exp\left(2\pi i \frac{x_2 \cdot \xi_1}{N}\right) \mathcal{F}_{N_2} u_{\xi_1}(x_2) \overline{\mathcal{F}_{N_1} v_{x_2}(\xi_1)}.$$

The decimation formula is used in the Fast Fourier Transform algorithm to reduce the computation of a Fourier transform to the computation of a Fourier transform over a small cyclic group [DPV08, Chapter 2]. We omit the proof of the decimation formula, but see [DJ17, Lemma 3.1]. Taking  $\text{supp } u, \text{supp } v \subseteq C_{k_1+k_2,A}$ , we get  $\text{supp } u_{\xi_1} \subseteq C_{k_1,A}$  and  $\text{supp } v_{x_2} \subseteq C_{k_2,A}$ . So by Cauchy-Schwarz,

$$\begin{aligned} |\langle \mathcal{F}_N u, v \rangle|^2 &\leq \left( \sum_{\xi_1 \in C_{k_1,A}} \sum_{x_2 \in C_{k_2,A}} |\mathcal{F}_{N_2} u_{\xi_1}(x_2)|^2 \right) \left( \sum_{\xi_1 \in C_{k_1,A}} \sum_{x_2 \in C_{k_2,A}} |\mathcal{F}_{N_1}^* v_{x_2}(\xi_1)|^2 \right) \\ &\leq \|1_{C_{k_2,A}} \mathcal{F}_{N_2} 1_{C_{k_2,A}}\|^2 \|u\|^2 \|1_{C_{k_1,A}} \mathcal{F}_{N_1} 1_{C_{k_1,A}}\|^2 \|v\|^2, \end{aligned}$$

which proves (3.2).

In particular, if there exists  $\varepsilon_1 > 0$  and a scale  $L^{-k}$  such that

$$\|1_{C_{k,A}} \mathcal{F}_N 1_{C_{k,A}}\| \leq (1 - \varepsilon_1) N^{-\delta - \frac{d}{2}} \quad (3.3)$$

then by scale-invariance this estimate holds for scale  $L^{-1}$ , and by (3.2),

$$\|1_{C_{k,A}} \mathcal{F}_N 1_{C_{k,A}}\| \leq \|1_{C_{1,A}} \mathcal{F}_{L^{-1}} 1_{C_{1,A}}\|^k \leq (1 - \varepsilon_1)^k N^{-\delta - \frac{d}{2}}$$

and the claim follows with

$$\varepsilon_0 := -\log_L(1 - \varepsilon_1).$$

So let's suppose that (3.3) fails, so at every  $L$ -adic scale,

$$\|1_{C_{k,A}} \mathcal{F}_N 1_{C_{k,A}}\| = N^{-\delta - \frac{d}{2}}.$$

However, we have

$$\|1_{C_{k,A}} \mathcal{F}_N 1_{C_{k,A}}\| \leq \|1_{C_{k,A}} \mathcal{F}_N 1_{C_{k,A}}\|_{HS} = \frac{(\text{card } A)^k}{N^{d/2}} = N^{-\delta - \frac{d}{2}}$$

where  $\|\cdot\|_{HS}$  denotes the Hilbert-Schmidt norm (the  $\ell^2$  norm of the vector of generalized eigenvalues) and  $\|\cdot\|$  denotes the operator norm (the  $\ell^\infty$  norm on generalized eigenvalues).

Thus the  $\ell^2$  and  $\ell^\infty$  norms of eigenvalues are the same, so  $P = 1_{C_{k,A}} \mathcal{F}_N 1_{C_{k,A}}$  has rank 1. In particular, all  $2 \times 2$  minors of  $P$  vanish.

However,

$$P_{x,\xi} = N^{-d/2} 1_{C_{k,A}}(x) 1_{C_{k,A}}(\xi) \exp\left(2\pi i \frac{x \cdot \xi}{N}\right)$$

so for every  $x, x', \xi, \xi' \in C_{k,A}$ ,

$$\det \begin{bmatrix} \exp(2\pi i x \cdot \xi/N) & \exp(2\pi i x' \cdot \xi/N) \\ \exp(2\pi i x \cdot \xi'/N) & \exp(2\pi i x' \cdot \xi'/N) \end{bmatrix} = 0$$

or equivalently,

$$\exp\left(2\pi i \frac{\langle x - x', \xi - \xi' \rangle}{N}\right) = 1.$$

In particular  $x - x'$  must be orthogonal to  $\xi - \xi'$  modulo  $N$ , a contradiction. Therefore there is a scale at which (3.3) holds.  $\square$

Let's recap the strategy here:

- (1) By working in the right norm, we witness the part of the Patterson-Sullivan exponent that we're trying to beat trivially, and we just need to get an improvement in that norm.
- (2) We discretize the fractal, and make the discretization so self-similar that if we get a multiplicative gain at one scale, we get an exponential gain at all scales.
- (3) If we cannot get a multiplicative gain at one scale, then the behavior of the fractal becomes highly constrained, and violates the nonorthogonality assumption.

These three steps are exactly the steps that work in the general case! But they become a lot more technical, essentially because a general Ahlfors-David set is only self-similar on average, rather than the perfect self-similarity of the arithmetic Cantor set.

Side remark: There's lots of interesting problems which have only been addressed in the self-similar case. For example we have [EH21] the following fractal uncertainty principle, which is much stronger than the Dyatlov–Jin uncertainty principle with overwhelming probability:

**Theorem 3.4.** *Let  $d = 1$  and let  $A \subseteq [L]$  be drawn at random. Then with probability  $\leq 3L \exp(-\frac{L^{4\varepsilon}}{64})$ ,*

$$\log_N \|1_{C_{k,A}} \mathcal{F}_N 1_{C_{k,A}}\| \leq -\max\left(0, \frac{1}{2} - \frac{3}{4}\delta - \varepsilon\right). \quad (3.4)$$

So it is natural to conjecture that for a “generic” Ahlfors-David set the improved bound (3.4) holds. To my knowledge even formulating this precisely is wide open.

#### 4. THE DYATLOV–JIN UNCERTAINTY PRINCIPLE

Let's prove the Dyatlov–Jin uncertainty principle.

**Definition 4.1.** A compact set  $X$  is *nonorthogonal* (to itself) on scales  $[h, 1]$  if for every  $x_0, \xi_0 \in X$  and  $h < r_1, r_2 \leq 1$  there exist  $x_1, x_2 \in B(x_0, r_1)$  and  $\xi_1, \xi_2 \in B(\xi_0, r_2)$  such that

$$|(x_1 - x_2, \xi_1 - \xi_2)| \gtrsim r_1 r_2.$$

This condition is also called the *nonconcentration property* or the *local nonintegrability condition* (LNIC).

**Example 4.2.** Lots of sets are nonorthogonal:

- (1) If  $d = 1$ ,  $\delta > 0$ , and  $X$  is  $\delta$ -regular, then  $X$  is nonorthogonal.
- (2) If  $\Gamma$  is a Zariski dense convex cocompact subgroup of  $SO^+(1, d + 1)$ , then  $\Lambda_\Gamma$  is nonorthogonal.
- (3) If  $X$  is nonorthogonal, then  $X_h$  is nonorthogonal on scales  $[h, 1]$ .
- (4) One may check by hand, using self-similarity, that a Sierpinski carpet is nonorthogonal.

**Theorem 4.3.** *Let  $X$  be nonorthogonal on scales  $[h, 1]$ , and suppose that  $X$  is the support of a doubling measure  $\mu$  (doubling on scales  $[h, 1]$ ). Let*

$$B_h f(x) := \int_X e^{ix \cdot \xi/h} f(\xi) d\mu(\xi).$$

Then there exists  $\varepsilon_0 > 0$  such that

$$\|B_h\|_{L^2(\mu) \rightarrow L^2(\mu)} \lesssim h^{\varepsilon_0}.$$

We apply this theorem with  $X$  replaced by  $X_h$  and  $\mu(E) := h^{\delta-d}|X \cap E|$ , which is a doubling measure if  $X$  is  $\delta$ -regular:

**Corollary 4.4.** *If  $X$  is nonorthogonal and  $\delta$ -regular,  $\delta \in (0, d/2]$ , then  $X$  satisfies the fractal uncertainty principle.*

**4.1. Discretization of compactly supported doubling measures.** Let  $X$  be a compact set, which we will think of as equipped with a doubling measure  $\mu$ . Also fix a parameter  $L \geq 3$ .

**Definition 4.5.** A *discretization* of a compact set  $X \subseteq \mathbf{R}^d$  is a set  $V(X) = \{V_n(X) : n \in \mathbf{Z}\}$  of sets, where  $V_n(X)$  is a set of subsets of  $\mathbf{R}^d$ , called *tiles*, such that:

- (1)  $X = \bigcup \{I \cap X : I \in V_n(X)\}$  and the union is disjoint,
- (2) For every  $I \in V_n(X)$  we have  $I = \bigcup_k I_k$  for some  $I_k \in V_{n+1}(X)$ .

In other words, the discretization turns the set  $X$  into the set of paths through a tree, and the set of the  $n$ th level of vertices is  $V_n(X)$ . If our discretization is “good” and  $\mu$  is doubling, then there should be some  $f(L)$  such that the measure of a child of a tile  $I$  should be  $\gtrsim f(L)\mu(I)$ . For example if  $\mu$  is a  $\delta$ -regular measure then  $f(L) = L^{-\delta}$ .

**Example 4.6.** The *standard discretization*  $V^0(X)$  is defined by letting  $V_n^0(X)$  be the set of all  $L$ -adic cubes of side length  $L^{-n}$  which intersect  $X$ . However, this discretization is not very useful for our purposes because it doesn’t “see” the doubling nature of  $\mu$ . For example, we could choose  $X = [L^{-1}, 1]$  and  $\mu$  Lebesgue measure on  $X$ , then for  $L \gg 1$ ,  $[0, 1]$  has measure  $\sim 1$ , but its child  $[0, L^{-1}]$  has measure zero, despite intersecting  $X$ .

**Example 4.7.** If  $X$  is an arithmetic Cantor set, then  $X$  comes with its own natural choice of discretization:  $C_{n,A}$  specifies a set of  $L$ -adic cubes of side length  $L^{-n}$ , so let  $V_n(X)$  be the set of all  $L$ -adic cubes specified by  $C_{n,A}$ . Note that this is just the standard discretization, but it witnesses that  $\mu$  is doubling, since each cube has measure  $L^\delta$  times the measure of its parent.

As it turns out, we can modify the standard discretization to get a measure for which the above counterexample on the standard discretization fails:

**Proposition 4.8.** *Let  $L$  be a multiple of  $10^3$ , and  $X$  a compact set. There exists a discretization  $V(X)$  such that for every tile  $I \in V_n(X)$ , there exists an  $L$ -adic cube  $I^0 \in V_n^0(X)$  such that*

$$I^0(1 - L^{-2/3}) \subseteq I \subseteq I^0(1 + L^{-2/3}),$$

and there exists a point  $x_0 \in I \cap X$  such that

$$\text{dist}(x_0, \partial I) \geq \frac{1}{10} L^{-2/3-n}.$$

*Proof.* We'll prove this for  $d = 1$  for simplicity.

Let's do it at scale  $L^{-n}$  first. Let  $I \in V_n^0(X)$ . We want to modify  $I$  to be a tile  $T$  so that there exists  $x_0 \in X \cap T$  such that

$$\text{dist}(x_0, \partial T) \geq \frac{1}{5} L^{-2/3-n}. \quad (4.1)$$

To do this, we start with setting  $T(I) := I$  for every  $I$ . We say that  $T$  is a *predator* if (4.1) holds, *dead* if  $X \cap T$  is empty, and otherwise *prey*.

We now iterate over all prey at scale  $L^{-n}$ . If  $T$  is prey, all  $x \in T \cap X$  are within  $L^{-2/3-n}/5$  of the boundary. If  $x$  is close to an adjacent predator  $S$ , we feed a subinterval of  $T$  of size  $L^{-2/3}/2$  to  $S$ , so that on that side,  $T$  no longer meets  $X$ . Otherwise,  $x$  is close to prey or a dead tile, so we feed a subinterval of size  $L^{-2/3}/2$  of the prey or dead tile to  $T$ . Then either  $T$  is a predator (because it ate one of its neighbors) or it is dead (its neighbors ate it). This operation created no new prey, and removed at least one prey, so this process will stop.

After this procedure is complete, every tile is either a predator or dead. Let  $V_n(X)$  be the set of predators. Since  $L$  is a multiple of  $10^3$  (in particular,  $L^{2/3}/10$  is an integer), it only moved entire children of each tile, so the tree structure  $V_n^0(X)$  is preserved. It never added or removed more than  $L^{-2/3}/2$  of the original interval to get the tile, plus  $L^{-5/3}/2$  from the previous scale (which is negligible since  $L \geq 10^3$ ).

If  $d \geq 2$ , we have notions of types of predator, where type  $k$  means that the tile has a problem with its  $k$ -boundary. Then we induct backwards, eliminating tiles of type  $d - 1, \dots, 1, 0$  as above. (You pick up some small losses doing this, which is why the gain is of size  $L^{-n-2/3}/10$  rather than  $L^{-n-2/3}/5$ ). We omit the details, because this is technical but the basic idea is already there for  $d = 1$ .  $\square$

Given  $I \in V_n(X)$ , we put a probability measure on the set  $\{I_a : a \in A\}$  of children of  $I$ , namely  $\text{Pr}(a) := \mu(I_a)/\mu(I)$ . Let  $D_\mu$  be the doubling constant of  $\mu$ , namely for every cube  $I$ ,

$$\mu(2I) \leq D_\mu \mu(I).$$

**Lemma 4.9.** *For any  $a$ ,*

$$\text{Pr}(a) \geq D_\mu^{-\log_2(3L)}.$$

*Proof.* By definition of  $V(X)$ , there exist cubes  $\tilde{I} \supseteq I$  and  $\tilde{I}_a \subseteq I_a$  such that  $\tilde{I} \subseteq 3L\tilde{I}_a$ . Then

$$\mu(I) \leq \mu(\tilde{I}) \leq D_\mu^{-\log_2(3L)} \mu(\tilde{I}_a) \leq D_\mu^{-\log_2(3L)} \mu(I_a). \quad \square$$

**Proposition 4.10.** *Suppose that  $X$  is the nonorthogonal support of a compactly supported doubling measure  $\mu$  and  $X$  is nonorthogonal. Given tiles  $I \in V_n(X)$  and  $J \in V_m(X)$ , draw children  $I_a, I_{a'}$  and  $J_b, J_{b'}$  independently at random, and let  $x_a, x_{a'}, \xi_b, \xi_{b'}$  be in their respective tiles. The probability that*

$$|(x_a - x_{a'}) \cdot (\xi_b - \xi_{b'})| \sim L^{-n-m-4/3}$$

and

$$L^{n+2/3}|x_a - x_{a'}|, L^{m+2/3}|\xi_b - \xi_{b'}| \sim 1$$

(meant in the limit as  $L \rightarrow \infty$ ) is at least

$$\rho := D_\mu^{-4 \log_2(2L)}. \quad (4.2)$$

*Proof.* By definition of  $V(X)$  and the fact that  $L \geq 10^3$ , there exist  $\underline{x} \in X \cap I$  and  $\underline{\xi} \in X \cap J$  which are  $\sim L^{-n-2/3}$  or  $L^{-m-2/3}$ -interior to their tiles. By nonorthogonality there exist  $\tilde{x}_a, \tilde{x}_{a'} \in I \cap X$  of distance  $\sim L^{-n-2/3}$  to  $\underline{x}$ , and  $\tilde{\xi}_b, \tilde{\xi}_{b'} \in Y \cap J$  of distance  $\sim L^{-m-2/3}$  to  $\underline{\xi}$ , such that

$$|(\tilde{x}_a - \tilde{x}_{a'}) \cdot (\tilde{\xi}_b - \tilde{\xi}_{b'})| \gtrsim L^{-n-m-4/3}.$$

In the other direction, Cauchy-Schwarz gives

$$\begin{aligned} |(\tilde{x}_a - \tilde{x}_{a'}) \cdot (\tilde{\xi}_b - \tilde{\xi}_{b'})| &\leq |\tilde{x}_a - \tilde{x}_{a'}| \cdot |\tilde{\xi}_b - \tilde{\xi}_{b'}| \\ &\leq \|\tilde{x}_a - \underline{x}\| + \|\tilde{x}_{a'} - \underline{x}\| \cdot \|\tilde{\xi}_b - \underline{\xi}\| + \|\tilde{\xi}_{b'} - \underline{\xi}\| \\ &\lesssim L^{-n-m-4/3}. \end{aligned}$$

In particular we have  $|\tilde{x}_a - \tilde{x}_{a'}| \sim L^{-n-2/3}$  and  $|\tilde{\xi}_b - \tilde{\xi}_{b'}| \sim L^{-m-2/3}$ . Let  $I_a$ , etc be the child containing  $x_a$ . If  $L$  is large then the diameter of  $I_a$  is much smaller than  $L^{-n-2/3}$  so we have  $|x_a - x_{a'}| \sim L^{-n-2/3}$  and similarly for  $\xi_b - \xi_{b'}$ , and similarly for the bounds on the dot product. We omit the details of this computation which is basically just two pages of the reverse triangle inequality.

So there exist  $a, a', b, b'$  in which the desired estimates hold. However

$$\Pr(a \cap a' \cap b \cap b') = \Pr(a) \Pr(a') \Pr(b) \Pr(b') \geq D_\mu^{-4 \log_2(3L)} = \rho. \quad \square$$

**4.2. Getting a gain at one scale.** Let  $K = -\log_L h$ . We assume that  $h$  is a reciprocal integer, and  $L$  is a large multiple of  $10^3/h$ , so  $K \geq 0$  is integral. In particular, we fix a scale  $L^{-n}$ , and define  $m$  by  $n + m + 1 = K$ . Let  $I \in V_n(X)$ ,  $J \in V_m(X)$ . We set

$$F_J(x) = \frac{1}{\mu(J)} \int_J e^{ix \cdot (\xi - \xi_J)/h} f(\xi) d\mu(\xi)$$

where  $f \in L^2(\mu)$  is given, and  $\xi_J$  is the center of the cube that we used to construct the tile  $J$ . We introduce the norms for  $\theta \in (0, 1)$ ,

$$\|F\|_{C_\theta(I)} := \max(\|F\|_{C^0(I)}, \theta \operatorname{diam}(I) \|\nabla F\|_{C^0(I_{conv})})$$

where  $I_{conv}$  is the convex hull of the tile  $I$ . Then for

$$\Psi_b(x) := \frac{x \cdot (\xi_J - \xi_{J_b})}{h},$$

we have (for  $\theta := \frac{1}{4}$  and  $L \gg 1$ )

$$\|e^{i\Psi_b} F\|_{C_\theta(I_a)} \leq \|F\|_{C_\theta(I)} \quad (4.3)$$

which follows because

$$\|\nabla \Psi_b\|_{C^0(I)} \leq \frac{\operatorname{diam}(J)}{h}.$$

This makes the norms  $C_\theta(I)$  useful for getting a gain at every scale. The point is that the  $C_\theta(I)$  norms are scale-invariant since they have a  $\operatorname{diam}(I)$  on the derivative term, while the usual  $C^1(I)$  norm is dimensionally inconsistent, so we can't use it for induction on scale.

We have the following gain at one scale. (Note that in light of (4.2)  $1/\varepsilon_1$  is actually polynomial in  $L$ !)

**Proposition 4.11** (Dolgopyat’s inequality). *Draw  $b$  at random. If  $L \gg 1$  and  $\varepsilon_1 := \rho/L$  then*

$$\|F_J\|_{C_\theta(I)}^2 \leq (1 - \varepsilon_1) \mathbf{E} \|F_{J_b}\|_{C_\theta(I)}^2.$$

Why is this result called *Dolgopyat’s inequality*? The first time a result appeared like this was in a paper of Dolgopyat [Dol98]. The ingredients are some sort of fractal or symbolic-dynamical structure, which satisfies some sort of spacing condition like LNIC, and a phase function which is interpreted as “oscillating much faster” in some sense than the function it is multiplied with. In particular, if Dolgopyat’s inequality fails, then it is because we have “almost equality” and so  $F_J$  cannot be less than the sum of its parts. But we have oscillation so this is absurd.

*Proof.* First note that  $F_J = \mathbf{E} e^{i\Psi_b} F_{J_b}$ . By Cauchy-Schwarz and (4.3) we have

$$\|F_J\|_{C_\theta(I_a)}^2 \leq (\mathbf{E} \|F_{J_b}\|_{C_\theta(I)}^2)^2 \leq \mathbf{E} \|F_{J_b}\|_{C_\theta(I)}^2 =: R.$$

Taking expectations we get

$$\sigma^2 := \mathbf{E} \|F_{J_b}\|_{C_\theta(I)}^2 - \mathbf{E} \|F_J\|_{C_\theta(I_a)}^2 \geq \mathbf{E} \|F_{J_b}\|_{C_\theta(I)}^2 - (\mathbf{E} \|F_{J_b}\|_{C_\theta(I)})^2 = \text{Var} \|F_{J_b}\|_{C_\theta(I)}.$$

We want to show that  $\sigma^2 \geq \varepsilon_1 R$ . So let’s suppose that’s not true, hence  $\sigma^2 < \varepsilon_1 R$ . Also let  $F_{ab} = F_{J_b}(x_a)$ ,  $\omega_{ab} = \Psi_b(x_a)$ ,  $f_{ab} = e^{i\omega_{ab}} F_{ab}$ . Then  $\mathbf{E} |F_{ab}|^2 \leq R$ .

Let’s draw  $a, a', b'$  independently of each other and  $b$ . By our contradiction assumption,  $\|F_{J_b}\|_{C_\theta(I)}$  is very nearly independent of  $b$ . We claim that the same holds for  $f_{ab}$ , which will lead to a contradiction when we condition on the event of Proposition 4.10 which implies that  $f_{ab}, f_{a'b'}$  are uncorrelated.

To make this precise, we first bound

$$\theta \|\nabla(e^{i\Psi_b} F_{J_b})\|_{C^0} \text{diam } I_a \leq \frac{\|F_{J_b}\|_{C_\theta(I)}}{2}.$$

From the definition of  $C_\theta(I_a)$  and the triangle inequality,

$$\|F_J\|_{C_\theta(I_a)} \leq \max \left( \|F_J\|_{C^0(I_a)}, \frac{1}{2} \mathbf{E} \|F_{J_b}\|_{C_\theta(I)} \right).$$

Also

$$\|F_J\|_{C^0(I_a)}^2 \leq \|F_J\|_{C_\theta(I_a)}^2 \leq R$$

and

$$\frac{1}{2} \mathbf{E} \|F_{J_b}\|_{C_\theta(I)}^2 \leq \frac{R}{2} \leq R + \|F_J\|_{C^0(I_a)},$$

so

$$\|F_J\|_{C_\theta(I_a)}^2 \leq \frac{1}{2} \left( R + \|F_J\|_{C^0(I_a)}^2 \right).$$

After taking expectations, we get

$$\mathbf{E} \|F_J\|_{C^0(I_a)}^2 \geq 2 \mathbf{E} \|F_J\|_{C_\theta(I_a)}^2 - R = R - 2\sigma^2.$$

Therefore, since  $|F_J(x_a)| = \|F_J\|_{C^0(I_a)}$ ,

$$|\mathbf{E} f_{ab}|^2 = \mathbf{E} |F_J(x_a)|^2 \geq R - 2\sigma^2.$$

Since

$$\mathbf{E} |F_{ab}| = \mathbf{E} |f_{ab}| \geq |\mathbf{E} f_{ab}|,$$

a Taylor expansion of the square root now gives

$$\mathbf{E} |F_{ab}| \geq (1 - 2\varepsilon_1)\sqrt{R}. \quad (4.4)$$

Moreover, by our contradiction assumption,

$$\begin{aligned} \mathbf{E} |F_{ab}|^2 &= \mathbf{E} |f_{ab}|^2 = |\mathbf{E} f_{ab}|^2 + \text{Var} f_{ab} \\ &\geq R - 2\sigma^2 + \text{Var} f_{ab}. \end{aligned}$$

Rearranging, we obtain

$$\text{Var} f_{ab} \leq \mathbf{E} |F_{ab}|^2 - (1 - 2\varepsilon_1)R.$$

Then we conclude

$$\text{Var} f_{ab} \leq \mathbf{E} |F_{ab}|^2 - (1 - 2\varepsilon_1)R \leq 2\sigma^2. \quad (4.5)$$

So at this point we have an unconditional bound on  $\text{Var} f_{ab}$ . Let's get some conditional bounds on related quantities that contradict each other. We know by Cauchy-Schwarz that  $\mathbf{E} \|F_{J_b}\|_{C_\theta(I)} \leq R^{1/2}$ , so by Chebyshev

$$\Pr(\|F_{J_b}\|_{C_\theta(I)} \leq 2\sqrt{R}) > 1 - \varepsilon_1.$$

We use this and (4.5) to get

$$\begin{aligned} \mathbf{E}(|f_{ab} - f_{ab'}|^2 | \max(\|F_{J_b}\|_{C_\theta(I)}, \|F_{J_{b'}}\|_{C_\theta(I)}) \leq 2\sqrt{R}) &\leq \sigma^2 + \mathbf{E}(|f_{ab} - f_{ab'}|^2) \\ &\leq \sigma^2 + 2\text{Var} f_{ab} \leq 5\sigma^2 \end{aligned}$$

hence

$$\mathbf{E}(|f_{ab} - f_{ab'}|^2 | \max(\|F_{J_b}\|_{C_\theta(I)}, \|F_{J_{b'}}\|_{C_\theta(I)}) \leq 2\sqrt{R}) \lesssim \sigma^2. \quad (4.6)$$

Let  $S$  be the intersection of the above event, with the event of Proposition 4.10, so (since  $\varepsilon_1 \ll \rho$ )  $\Pr(S) \gtrsim \rho$ . By the mean value theorem and the definition of  $C_\theta$ ,

$$|F_{ab} - F_{a'b}| \leq \frac{2R^{1/2}}{\theta} L^{H(I)} |x_a - x_{a'}|.$$

In particular if  $S$  holds,

$$|F_{ab} - F_{a'b}| \lesssim R^{1/2} L^{-2/3}. \quad (4.7)$$

Here's where the contradiction comes in. Let  $\tau := (x_a - x_{a'}) \cdot (y_b - y_{b'}) = \omega_{ab} - \omega_{ab'} - \omega_{a'b} + \omega_{a'b'}$ . If  $S$  holds, then

$$|e^{i\tau} - 1| \geq |\tau| \gtrsim L^{-1/3}$$

but on the other hand, the  $L^2$  triangle inequality gives

$$\mathbf{E}(|(e^{i\tau} - 1)F_{ab}|^2 | S) \leq 4 \mathbf{E}(|F_{ab} - F_{a'b}|^2 + |F_{a'b'} - F_{ab'}|^2 + |f_{ab} - f_{ab'}|^2 + |f_{a'b'} - f_{a'b}|^2 | S)$$

where the first two terms are controlled by (4.7) as  $\lesssim R/L^{4/3}$  and the latter are controlled by (4.6) and  $\Pr(S) \gtrsim \rho$  as  $\lesssim \sigma^2/\rho$ . Summing up,

$$\mathbf{E}(|F_{ab}|^2 | S) \lesssim \frac{R}{L^{2/3}} + \frac{L^{2/3}\sigma^2}{\rho}. \quad (4.8)$$

On the other hand

$$\begin{aligned} \Pr(|F_{ab}|^2 \leq R/5) &\leq \Pr(|F_{ab}| \leq \mathbf{E} |F_{ab}| - R^{1/2}/2) \\ &\lesssim R \text{Var} |F_{ab}| = R \text{Var} f_{ab} \leq 2R\sigma^2 \lesssim \varepsilon_1. \end{aligned}$$

So

$$\Pr(|F_{ab}|^2 \leq R/5|S) \leq \frac{\Pr(|F_{ab}|^2 \leq R/5)}{\Pr(S)} \lesssim \frac{\varepsilon_1}{\rho} = \frac{1}{L}.$$

By Markov,

$$\mathbf{E}(|F_{ab}|^2|S) \geq \frac{R}{5} \Pr(|F_{ab}|^2 \geq R/5|S) = \frac{R}{5} \left(1 - \frac{1}{L}\right) \gtrsim R.$$

Combining this with (4.8) and our contradiction assumption, we get

$$R \lesssim \frac{R}{L^{2/3}} + \frac{L^{2/3}\sigma^2}{\rho} < \frac{R}{L^{2/3}} + \frac{R}{L^{1/3}}$$

which is absurd if  $L \gg 1$ .  $\square$

**4.3. Induction.** Let  $E_J : V_{K-H(J)} \rightarrow \mathbf{R}$  be defined by  $E_J(I) := \|F_J\|_{C_\theta(I)}$ . Here  $L^{-K} = h$ . We put a measure on  $V_n$  by  $\mu(\{I\}) = \mu(I)$ . For the base case we take  $H(J) = K$ , and use Cauchy-Schwarz

$$\begin{aligned} |\nabla F_J(x)| &= \frac{1}{\mu(J)} \int_J i\partial_x \Psi_J(x, y) \exp(i(\Psi_J(x, y))) f(x, y) \\ &\leq \frac{\text{diam } J}{h\mu(J)} \|f\|_{L^2(J)}. \end{aligned}$$

We also have

$$\|F_J\|_{C^0} \leq \frac{\|f\|_{L^2(J)}}{\sqrt{\mu(J)}}.$$

In particular, since  $\text{diam } J \sim h$ ,

$$E_J(I) \lesssim \frac{\|f\|_{L^2(J)}}{\sqrt{\mu(J)}}.$$

Taking  $L^2$  norms of both sides in the variable  $I$  using the measure on  $V_{K-H(J)}$  and using  $\mu(X) \lesssim 1$ ,

$$\|E_J\|_{L^2}^2 \lesssim \frac{\|f\|_{L^2(J)}^2}{\mu(J)},$$

and taking  $L^2$  norms of both sides in Dolgopyat's inequality, we get for  $H(J) < K$ ,

$$\|E_J\|_{L^2}^2 \leq (1 - \varepsilon_1) \frac{\mu(J)}{\mu(J_b)} \mathbf{E} \|E_{J_b}\|_{L^2}^2.$$

So by induction backwards on  $H(J)$ , we get for  $H(J) = 0$ ,

$$\|E_J\|_{L^2}^2 \lesssim \frac{(1 - \varepsilon_1)^K}{\mu(J)} \|f\|_{L^2(J)}^2.$$

Also

$$\|B_h(1_J f)\|_{L^2}^2 \lesssim \|F_J\|_{L^2}^2 \leq \|E_J\|_{L^2}^2$$

so summing in the  $O(1)$  many  $J$  with  $H(J) = 0$  which it takes to cover  $X$ ,

$$\|B_h f\|_{L^2} \lesssim (1 - \varepsilon_1)^{K/2} \|f\|_{L^2}.$$

If we take  $\varepsilon_0 := 0.5 \cdot \log_L(1 - \varepsilon_1)$  then  $(1 - \varepsilon_1)^{K/2} = h^{\varepsilon_0}$ , which completes the proof of the Dyatlov–Jin uncertainty principle.

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