PROPERTIES OF FOURIER INTEGRAL OPERATORS

AIDAN BACKUS

1. Summary of the seminar so far

Let Λ be a closed conic Lagrange submanifold of $T^*X \setminus 0$, and let $n = \dim X$. In Yonah's talk we defined a quantization map

$$S_\rho^{m+n/4}(\Lambda, \Omega_{1/2} \otimes L) \to I_\rho^m(X, \Lambda)$$

where L is the Maslov line bundle (defined in Zhongkai's talk) of Λ and $\Omega_{1/2}$ is the half-density bundle on X. This map is an isomorphism modulo worse classes. So the principal symbol of an oscillatory integral in $I_\rho^m(X, \Lambda)$ is an element of $S_\rho^{m+n/4}(\Lambda, \Omega_{1/2} \otimes L)$ which is well-defined modulo worse classes.

Henceforth we will suppress all tensor products against $\Omega_{1/2}$, because every vector bundle that we care about will be tensored against $\Omega_{1/2}$. The idea is that we can integrate a half-density u against a test function invariantly. We're doing analysis (or quantum mechanics) so we don't care about $u(x)$, we just care about how u integrates against test functions.

Recall James' talk: If C is a homogeneous canonial relation from X to Y, then

$$C \subseteq (T^*X \setminus 0) \times (T^*Y \setminus 0)$$

is a closed conic Lagrange manifold, where the symplectic form on $T^*X \setminus T^*Y$ is given by $\sigma_X - \sigma_Y$, where σ_Z is the symplectic form on T^*Z. The Lagrange manifold C' obtained by multiplying by -1 in the fibers over Y satisfies the following: elements of $I_\rho^m(X \times Y, C')$ define Fourier integral operators $\mathcal{E}'(Y) \to \mathcal{D}'(X)$.

2. More preliminaries on half-densities

Lemma 2.1. $\Omega^{1/2}$ *is a trivial line bundle.*

Proof. Choose a Riemannian metric g on X. Then $|dV| = \sqrt{g}$ is a global volume density so Ω_1 is trivial. Since $\Omega_{1/2}$ is the tensor square root of Ω_1, the claim holds. □

Date: July 2021.

Now we recall the notion of Lie derivative for half-densities. In general, if a is a section of a vector bundle E and v is a vector field which induces a one-parameter group φ, we have

$$\mathcal{L}_v a = \frac{\partial}{\partial t}|_{t=0}\varphi_t^* a.$$

The idea: v is the velocity field of a fluid, a is a physical invariant of the fluid, and $\mathcal{L}_v a$ describes how a changes as the fluid flows. If E is a trivial line bundle we can write $a = ua_0$ for a_0 a nonvanishing section of E. Now $\mathcal{L}_v a_0$ is a scalar multiple of a_0, say $\mathcal{L}_v a_0 = fa_0$ for some smooth function f, so by the Leibniz rule

$$\mathcal{L}_v a = \frac{\partial u}{\partial v}a_0 + fua_0.$$

It remains to compute f.

Suppose that $E = \Omega^{1/2}$. Local coordinates x allow us to think of the domain as a subset of $\mathbf{R}^n$ so we get a notion of "boxes", a volume form dV, and the notion of a flux form, as well as the divergence $\operatorname{div} v = \sum_j \partial_{x_j} v_j$. Suppose that

$$a_0 \otimes a_0 = |dx_1 \wedge \cdots \wedge dx_n| = |dV|.$$

That is,

$$2(\mathcal{L}_v a_0) \otimes a_0 = \mathcal{L}_v |dV| \otimes a_0.$$

Lemma 2.2. $\mathcal{L}_v |dV| = \operatorname{div} v |dV|$.

Proof. Fix a small box B. Then

$$\iint_B \mathcal{L}_v |dV| = \iint_B \frac{\partial}{\partial t}|_{t=0}\varphi_t^*|dV| = \frac{\partial}{\partial t}|_{t=0} \iint_{(\varphi_t)_* B} |dV|$$

where we used the transformation law for densities. Moreover $(\varphi_t)_* B$ is contractible so as long as we choose it to be positively oriented,

$$\frac{\partial}{\partial t}|_{t=0} \iint_{(\varphi_t)_* B} |dV| = \frac{\partial}{\partial t}|_{t=0} \iint_{(\varphi_t)_* B} dV = \int_{\partial B} \operatorname{flux} v.$$

Here we used the rule for differentiating the integral of a moving region. By the divergence theorem,

$$\int_{\partial B} \operatorname{flux} v = \iint_B \operatorname{div} v \ dV = \iint_B \operatorname{div} v \ |dV|$$

as desired. □

Anyways,

$$(\mathcal{L}_v a_0) \otimes a_0 = \frac{1}{2} \operatorname{div} v a_0 \otimes a_0.$$

Multiplying both sides by a_0^{-1} (which is a $-1/2$-density) we get:

Theorem 2.3 (the derivative of a half-density)**.** *Suppose that a is a half-density written in local coordinates. Then*

$$\mathcal{L}_v a = \left(\frac{\partial}{\partial v} + \frac{1}{2}\operatorname{div} v\right) a$$

where the divergence is defined using the pullback of the flat metric on $\mathbf{R}^n$.

3. The parametrix of an elliptic operator

Definition 3.1. Let $a \in S_\rho^{m+n/4}(\Lambda, L)$ be a principal symbol of $A \in I_\rho^m(X, \Lambda)$. We say that A is *noncharacteristic* or *elliptic* at $\lambda \in \Lambda$ if $1/a \in S_\rho^{-m-n/4}(\Lambda, \Omega_{-1/2})$, at least near the fiber infinity of a conic neighborhood of λ.

Clearly the choice of principal symbol does not matter. Moreover, invertibility of the principal symbol map (Theorem 3.2.6 in Hörmander I) implies that $\lambda \in WF(A)$.

Recall that a Fourier integral operator is said to be a smoothing operator if it has a smooth Schwartz kernel.

Proposition 3.2 (local existence of parametrices)**.** *Let $C : T^*Y \setminus 0 \to T^*X \setminus 0$ be a homogeneous symplectomorphism, let $K \subseteq \operatorname{graph} C$ be a closed conic set, and let $A \in I_\rho^m(X \times Y, K')$ be elliptic at $((x_0, \xi_0), (y_0, \eta_0))$. Then there exists $B \in I_\rho^{-m}(Y \times X, (K^{-1})')$ which is a local inverse to A modulo smoothing operators in the sense that $(x_0, \xi_0) \notin WF(AB - 1)$ and $(y_0, \eta_0) \notin WF(BA - 1)$.*

Proof sketch. The idea is basically the same as for pseudodifferential operators. First show that asymptotic sums are well-defined, then use a Neumann series argument to show that if B_0 is the quantization of $1/a$ near infinity, a the principal symbol of A, then AB_0 is invertible modulo smoothing operators. □

Clearly we can glue together local inverses B to get a global inverse, if A is in fact globally elliptic.

4. Subprincipal symbols

By expressing pseudodifferential calculus in terms of Lie derivatives, let us show that the principal symbol is not the only part of the full symbol of a pseudodifferential operator which is well-defined.

Proposition 4.1 (existence of subprincipal symbols)**.** *Let $P \in L_\rho^m$ and let p be the full symbol of P in some local coordinates x. Let*

$$c = p - (2i)^{-1} \sum_j \frac{\partial^2 p}{\partial x_j \partial \xi_j}.$$

Then $c \mod S_\rho^{m+2(1-2\rho)}$ does not depend on the choice of coordinates.

Proof. To ease the notation let me do the case $\rho = 1$ as the general case is similar.

If x is a system of coordinates on X, and φ is a diffeomorphism, then we set

$$\varphi(x,\theta) = \sum_j \varphi_j(x)\theta_j.$$

Let w be a half-density. Then

$$e^{-i\varphi} P(we^{i\varphi}) \sim \sum_\alpha \frac{1}{\alpha!} p^{(\alpha)}(x,\varphi'_x) D_z^\alpha (w(z)e^{i\rho(x,z,\theta)})|_{z=x}$$

where

$$\rho(x,z,\theta) = \varphi(z,\theta) - \varphi(x,\theta) - \langle z-x, \varphi'_x(x,\theta)\rangle$$

(so $\rho(\cdot,\cdot,\theta)$ vanishes to second order at x) and $p^{(\alpha)}(x,\xi) = -iD_\xi^\alpha(x,\xi)$. This is nothing more than the change-of-variables formula for pseudodifferential operators. For $|\alpha| = 3$, $D_z^\alpha(w(z)e^{i\rho(x,z,\theta)})|_{z=x}$ is linear in θ so

$$D_z^\alpha(w(z)e^{i\rho(x,z,\theta)})|_{z=x} \in S^1$$

while clearly $p^{(\alpha)} \in S^{m-3}$, so

$$p^{(\alpha)}(x,\varphi'_x) D_z^\alpha(w(z)e^{i\rho(x,z,\theta)})|_{z=x} \in S^{m-2}.$$

Cutting off to $|\alpha| \leq 2$ we get

$$\begin{aligned} e^{-i\varphi}P(we^{i\varphi}) = p(x,\varphi'_x)w - i\sum_j \frac{\partial p}{\partial \xi_j}(x,\varphi'_x)\frac{\partial w}{\partial x_j} \\ + (2i)^{-1}\sum_{j,k} \frac{\partial^2 p}{\partial \xi_j \partial \xi_k}(x,\varphi'_x) w(x) \frac{\partial^2 \varphi}{\partial x_j \partial x_k} \mod S^{m-2}. \end{aligned}$$

Let

$$v = \left(\frac{\partial p}{\partial \xi_1}(x,\varphi'_x), \dots, \frac{\partial p}{\partial \xi_n}(x,\varphi'_x)\right)$$

so

$$\operatorname{div} v = \sum_j \frac{\partial^2 p}{\partial x_j \partial \xi_j}(x,\varphi'_x) + \sum_{j,k} \frac{\partial^2 p}{\partial \xi_j \partial \xi_k}(x,\varphi'_x)\frac{\partial^2 \varphi}{\partial x_j \partial x_k}.$$

In these coordinates we have $|dV|^{1/2} = 1$ so

$$\mathcal{L}_v w = \frac{\partial w}{\partial v} + \frac{1}{2}\operatorname{div} v.$$

Moreover

$$\sum_j \frac{\partial p}{\partial \xi_j}(x,\varphi'_x)\frac{\partial w}{\partial x_j} = \frac{\partial w}{\partial v}$$

so

$$\begin{aligned} e^{-i\varphi}P(we^{i\varphi}) &= p(x,\varphi_x')w + i\left(\frac{\partial w}{\partial v} + \frac{1}{2}\operatorname{div} v\right) - (2i)^{-1}\sum_j \frac{\partial^2 p}{\partial x_j \partial \xi_j}(x,\varphi_x')w \mod S^{m-2} \\ &= p(x,\varphi_x')w - (2i)^{-1}\sum_j \frac{\partial^2 p}{\partial x_j \partial \xi_j}(x,\varphi_x')w + i\mathcal{L}_v w \mod S^{m-2} \end{aligned}$$

or in other words

$$e^{-i\varphi}P(we^{i\varphi}) + i\mathcal{L}_v w = p(x,\varphi_x')w - (2i)^{-1}\sum_j \frac{\partial^2 p}{\partial x_j \partial \xi_j}(x,\varphi_x')w \mod S^{m-2}$$

and the left-hand side is invariantly defined. Therefore so is the right-hand side. □

Definition 4.2. Let $\rho = 1$ and $P \in L^m$ have homogeneous principal symbol p. If the full symbol of P in some coordinate system is $p + r$, $r \in S^{m-1}$, set

$$c = r - (2i)^{-1}\sum_j \frac{\partial^2 p}{\partial x_j \partial \xi_j} \in S^{m-1}.$$

Then c is called the *subprincipal symbol* of P.

The above proposition says that the subprincipal symbol does not depend on a coordinates and the full symbol q of P satisfies

$$q = p + \sum_j \frac{\partial^2 p}{\partial x_j \partial \xi_j} + c \mod S^{m-2}$$

in any coordinate system whatsoever. Here p is the order-m part and $c + \partial^2 p/(\partial x_j \partial \xi_j)$ is the order-$m-1$ part of q.

Subprincipal symbols allow us to fix a defect in a theorem from last week's talk (Hormander 1, Thm 4.3.3):

Example 4.3. Let $p(\xi,\eta) = \xi^2 + \eta$. This is a full symbol with principal symbol ξ^2 and subprincipal symbol η. If $a(\xi,\eta)$ is a smoothed out version of $1_{1\leq\xi^2\leq 2}$ and P, A are their quantizations (with $\varphi(x,y,\xi,\eta) = x\xi + y\eta$ of course), then $P = \partial_x^2 + \partial_y$ and A is a Littlewood-Paley projection which means that

$$PA \sim \partial_y A \in L_1^1$$

where $B \sim Q$ means that they have the same principal symbol. Thus the principal symbol of PA is $\eta a(\xi,\eta)$ which is linear at infinity. However, Thm 4.3.3 computes the principal symbol of PA viewed as an element of L_1^2, and thus only considers terms that are quadratic at infinity. Thus Thm 4.3.3 thinks that the principal symbol of PA is 0!

In what follows we write H_p for the Hamiltonian vector field of a symbol p. We will use the following hypothesis a lot so let's emphasize it:

Definition 4.4. Let $P \in L^m_1(X)$ with homogeneous principal symbol p. Suppose that $C \subseteq (T^*Y \setminus 0) \times (T^*X \setminus 0)$ is a homogeneous canonical relation such that $p| \operatorname{range} C = 0$. Then we say that P *degenerates* on C to order m.

Theorem 4.5 (principal symbols of degenerate products)**.** *Suppose that P degenerates on C to order m. Let c be the subprincipal symbol of P. If $A \in I^{m'}_\rho(X \times Y, C')$ has principal symbol $a \in S^{m'+(n_X+n_Y)/4}(C', L)$ then $PA \in I^{m+m'-\rho}_\rho(X \times Y, C')$ and the principal symbol of PA is*

$$\sigma(PA) = (c - i\mathcal{L}_{H_p})a.$$

Example 4.6. In our example $p(\xi, \eta) = \xi^2 + \eta$, $a(\xi, \eta) \approx 1_{1 \leq \xi^2 \leq 2}$ we get

$$\sigma(PA) = (\eta - i\mathcal{L}_{H_p})a = \eta a + 2i\xi \frac{\partial a}{\partial x} = \eta a.$$

This is exactly what we got through our back-of-the-napkin computation.

Proof. The proof is technical but it's more or less what you'd expect. To get rid of the unwanted top-order terms you write PA in a clever way where you can integrate by parts to put a derivative on the symbol of A.

Since C' is a Lagrange manifold it is generated by a phase function

$$\varphi(x, y, \xi, \eta) = \langle x, \xi \rangle + \langle y, \eta \rangle - H(\xi, \eta)$$

where H is homogeneous conic-near $(\xi_0, \eta_0) \in C'$ and x, y are suitable coordinates on a patch $\tilde{X} \times \tilde{Y}$. That is, we may write

$$Au(x) = \iiint_{\tilde{Y} \times \mathbf{R}^{n_X+n_Y}} e^{i\varphi(x,y,\xi,\eta)} a(x, y, \xi, \eta) u(y) \; dy \wedge d\xi \wedge d\eta$$

where $a \in S^{m'-(n_X+n_Y)/4}_\rho$ is supported conic-near $(H'_\xi, H'_\eta, \xi, \eta)|_{(\xi,\eta)=(\xi_0,\eta_0)}$.

If M is the critical manifold of φ, then

$$\begin{aligned} \iota : (T^*\tilde{X} \setminus 0) \times (T^*\tilde{Y} \setminus 0) &\to M \\ (\xi, \eta) &\mapsto (H'_\xi, H'_\eta, \xi, \eta) \end{aligned}$$

is a local diffeomorphism, by my talk. Also φ induces a trivialization of the Maslov line bundle L over $\tilde{X} \times \tilde{Y}$. In this trivialization, the principal symbol of A on C is $(\xi, \eta) \mapsto a(H'_\xi, H'_\eta, \xi, \eta)$. Using the local diffeomorphism ι and the fact that the support of a can be made arbitrarily small around M, we may view a as a function a_0 of (ξ, η) only. This shrinking only changes a by a term in $S^{\mu+1-2\rho}_\rho$ where

$$\mu = m' - 0.25(n_X + n_Y)$$

is the order of a.

Commuting P with the integral sign we get

$$PAu(x) = \iiint_{\tilde{Y}\times\mathbf{R}^{n_X+n_Y}} e^{i\varphi(x,y,\xi,\eta)}(p(x,\xi)+r(x,\xi))a_0(\xi,\eta)u(y)\ dy\wedge d\xi\wedge d\eta$$

where $p+r$ is the full symbol of P in $\tilde{Y}$. This is allowed because P only differentiates in the x variables and the only x variables are on $e^{i\varphi(x,y,\xi,\eta)}$. Applying a pseudodifferential operator to differentiate $e^{i\langle x,\xi\rangle}$ in x only multiplies $e^{i\langle x,\xi\rangle}$ by the symbol, which is where we get our formula from.

Our hypothesis on p localized to $\tilde{X}$ gives $p(H'_\xi,\xi)=0$, and p is m-homogeneous. So we can find p_j which is m-homogeneous on $\tilde{X}\times\mathbf{R}^{n_X+n_Y}$ such that

$$p(x,\xi) = \sum_j p_j(x,\xi,\eta)\frac{\partial\varphi}{\partial\xi_j}(x,y,\xi,\eta)$$

Namely we can take $p_j(\cdot,\eta)$ to be the derivative of p with respect to $x_j - \partial_{\xi_j}H = \partial_{\xi_j}\varphi$ and apply Taylor's formula, using the nondegeneracy of φ. (This is where we use the hypothesis that the product PA degenerates!) The choice of η does not matter. Thus

$$\begin{aligned} PAu(x) = &\iiint_{\tilde{Y}\times\mathbf{R}^{n_X+n_Y}} e^{i\varphi(x,y,\xi,\eta)}a_0(\xi,\eta) \\ &\left(r(x,\xi)+\sum_j p_j(x,\xi,\eta)\frac{\partial\varphi}{\partial\xi_j}(x,y,\xi,\eta)\right)u(y)\ dy\wedge d\xi\wedge d\eta. \end{aligned}$$

We are only interested in behavior at fiber-infinity so we might as well assume that $a_0=0$ near 0, that way when we integrate by parts we don't pick up any junk at the origin (since technically these integrals are over $\tilde{Y}\times(\mathbf{R}^{n_X}\setminus 0)\times(\mathbf{R}^{n_Y}\setminus 0)$, as we don't have any control over the functions at 0).

Integrating with parts in ξ_j and using the support property of a_0, and letting

$$b(x,y,\xi,\eta) = r(x,\xi)a_0(\xi,\eta) + i\sum_j \frac{\partial p_j}{\partial\xi_j}(x,\xi,\eta)a_0(\xi,\eta) + i\sum_j\frac{\partial a_0}{\partial\xi_j}(\xi,\eta)p_j(x,\xi,\eta),$$

we get

$$PAu(x) = \iiint_{\tilde{Y}\times\mathbf{R}^{n_X+n_Y}} e^{i\varphi(x,y,\xi,\eta)}b(x,y,\xi,\eta)u(y)\ dy\wedge d\xi\wedge d\eta.$$

Thus PA is a Fourier integral operator with phase φ and full symbol $b\in S^{m+\mu-\rho}_\rho$ on $\tilde{Y}$. We picked up the $-\rho$ term from the differentiation in ξ_j. Therefore $PA\in I^{m+m'-\rho}_\rho$. This corrects the naive calculation that $PA\in I^{m+m'}_\rho$.

Since $a - a_0 \in S_\rho^{\mu+1-2\rho}$, we can replace a_0 with a in the definition of b without changing its residue class modulo $S_\rho^{\mu+1-2\rho}$. With $x = H'_\xi, y = H'_\eta$ we get

$$\sum_j \frac{\partial a_0}{\partial \xi_j}(\xi,\eta) p_j(x,\xi,\eta) = -\sum_j p_j(x,\xi,\eta) \sum_k \frac{\partial a}{\partial x_k} p_j(x,\xi,\eta) \frac{\partial^2 H}{\partial \xi_k \partial \xi_j} + p_j(x,\xi,\eta) \sum_k \frac{\partial a}{\partial y_k} \frac{\partial^2 H}{\partial \eta_k \partial \xi_j} + p_j(x,\xi,\eta) \frac{\partial a}{\partial \xi_j}(x,\xi,\eta).$$

Also $\partial_{x_j} p = p_j$, $\partial_{\xi_k} p = -\sum_j p_j \partial_{\xi_j} \partial_{\xi_k} p$, $0 = -\sum_j p_j \partial_{\xi_j} \partial_{\eta_k} H$, we get

$$-\sum_j \frac{\partial a_0}{\partial \xi_j}(\xi,\eta) p_j(x,\xi,\eta) = \sum_j \frac{\partial p}{\partial \xi_j} \frac{\partial a}{\partial x_j} - \frac{\partial p}{\partial x_j} \frac{\partial a}{\partial \xi_j} = \{p, a\}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket on T^*M.

Taking the Lie derivative of the half-density a with respect to the Hamiltonian vector field H_p we get

$$\mathcal{L}_{H_p} a = -\{p, a\}|dV| - \frac{1}{2} \operatorname{div}_\xi H_p |dV|$$

where $|dV|$ is a fixed half-density. Now everything cancels and we get

$$b = -i\mathcal{L}_{H_p} a + (r - (2i)^{-1}) \sum_j \frac{\partial^2 p}{\partial x_j \partial \xi_j} \mod S_\rho^{\mu+1-2\rho}.$$

But $(r - (2i)^{-1}) \sum_j \frac{\partial^2 p}{\partial x_j \partial \xi_j}$ is the subprincipal symbol of P so we're done. □

Now let us solve the equation $PA = B$ for A, where B is a given Fourier integral operator and P is a given pseudodifferential operator. If the solution to $PA = B$ has principal symbols p, a, b then by the previous theorem,

$$b = ca - i\mathcal{L}_{H_p} a$$

where c is the subprincipal symbol of P. Also, since these are Fourier integral operators, what we really want is to find A such that $PA - B$ is smoothing.

Again we will repeatedly use a hypothesis so we give it a name that's not in Hormander.

Definition 4.7. Suppose that P degenerates on C to order m, $p = \sigma(P)$, and $b \in S_\rho^{m+m'-\rho+n/4}$. Suppose that for every $\mu \in \mathbf{R}$,

$$S_\rho^{m-1+\mu}(C) \subseteq H_p S_p^\mu(C).$$

Then we say that (p, b) is *solvable*.

Lemma 4.8 (inverting a degenenerate symbol)**.** *Suppose that* (p, b) *is solvable. Then there exists* $a \in S_\rho^{m'+n/4}(C', L)$ *such that* $b = ca - i\mathcal{L}_{H_p} a$.

Proof. Let ω be a nonvanishing global section of $\Omega_{1/2}$ which is homogeneous of degree 0. This exists since $\Omega_{1/2}$ is trivial. Suppose $b = b_0\omega$. Then we must solve the scalar equation

$$(c' - iH_p)a_0 = b_0$$

where $c' \in S_1^{m-1}$. Since (p, b) is solvable there exists $\gamma \in S_\rho^0$ such that $H_p\gamma = c'$. The imaginary-exponential of a bounded symbol is bounded, i.e. $e^{i\gamma} \in S_\rho^0$. Writing $a_0 = ie^{-i\gamma}a_1$, $b_0 = e^{-i\gamma}b_1$, we must solve

$$\{p, a_1\} = b_1$$

for a_1, which is possible since (p, b) is solvable. □

Theorem 4.9 (inverting a degenerate pseudodifferential operator)**.** *Suppose that* (p, b) *is solvable and* $B \in I_\rho^{m+m'-1}(X \times Y, C')$ *is the quantization of* B. *Then there exists* $A \in I_\rho^{m'}(X \times Y, C')$ *such that* $PA - B$ *is smoothing. Moreover, if* $b = \sigma(B)$ *and* $a \in S_\rho^{m'+n/4}(C, L)$ *satisfies* $b = ca - i\mathcal{L}_{H_p} a$, *then in fact*

$$\sigma(A) = a \mod S_\rho^{m'+n/4+2-3\rho}(C, L).$$

Proof. By the lemma we can solve for a. Let A_0 be its quantization. Inductively set

$$B_{j+1} = B_j - PA_j.$$

Then we obtain $A_j \in I_\rho^{m'-j(3\rho-2)}(X \times Y, C')$ and $B_j \in I_\rho^{m+m'-1-j(3\rho-2)}(X \times Y, C')$, by the previous theorem. (The conclusion of that theorem is where the weird 2/3 factor comes from). Summing up these inductive equations we get

$$P \sum_{j \leq k} A_j = B_0 - B_{k+1}.$$

Let $A \sim \sum_j A_j$. This is possible since $3\rho > 2$ so

$$\lim_{j \to \infty} m' - j(3\rho - 2) = -\infty.$$

In particular B_{k+1} converges to a smoothing operator as $k \to \infty$. Thus $PA = B_0 \mod I^{-\infty}$. □

Corollary 4.10. *We can also solve* $AP = B$ *for* A *under the same hypotheses.*

Proof. Consider the adjoint equation. □

5. Regularity of Fourier integral equations

Let $H_s(X)$ be the space of distributions u such that for every properly supported $A \in L^s_1$, $Au \in L^2_{loc}(X)$. Recall from Mitchell's talk (Hormander 1, Crly 2.2.3) that if $B \in L^m_1$, then B maps $H_s \to H_{s-m}$.

Theorem 5.1 (regularity of Fourier integral equations)**.** *$I^m_\rho(X, \Lambda) \subseteq H_s(X)$ iff $m + n/4 + s < 0$. Moreover, if $u \in I^m_\rho(X, \Lambda)$ is elliptic somewhere and $m + n/4 + s \geq 0$ then $u \notin H_s(X)$.*

Of course if u is not elliptic anywhere, then u isn't "really" of order m.

Proof. Let $u \in I^m_\rho(X, \Lambda)$ and suppose $WF(u)$ is a small conic neighborhood Γ of $(x_0, \xi_0) \in \Lambda$. The claim is local, and if it's true when $s = 0$ then applying an elliptic operator $A : H_s \to L^2_{loc}$ we obtain it for $s \neq 0$ as well. This is true because A is an isomorphism modulo smoothing operators. So we must show that $u \in L^2_{loc}$ if $m + n/4 < 0$, and $u \notin L^2_{loc}$ if $m + n/4 \geq 0$ and u is elliptic at (x_0, ξ_0).

Let χ be a homogeneous canonical transformation from a conic neighborhood of (x_0, ξ_0) to a conic neighborhood of $(0, \eta_0) \in T^*\mathbf{R}^n \setminus 0$ and let K be a conic neighborhood of $(x_0, \xi_0, 0, \eta_0)$ in the graph of χ. Then (local existence of paramatrices) says that there exist $A \in I^0_1(X \times \mathbf{R}^n, K')$ and $B \in I^0_1(\mathbf{R}^n \times X, (K^{-1})')$ such that $(x_0, \xi_0) \notin WF(AB-1)$. Here B is a Fourier integral operator from X to $\mathbf{R}^n$ and A is a local parametrix to B. Shrinking Γ, we may assume that $WF(AB - 1) \cap \Gamma$ is empty. Then $AB - 1$ is smoothing on Γ, i.e. $(AB - 1)u \in C^\infty$. By Hormander 1, Thm 4.3.1 (from last week's talk), which says that properly supported operators in I^0 send L^2_{loc} to L^2_{loc}, implies

$$u \in L^2_{loc}(X) \implies Bu \in L^2_{loc}(\mathbf{R}^n) \implies ABu \in L^2_{loc}(X) \implies u \in L^2_{loc}(X).$$

Thus $u \in L^2_{loc}(X)$ iff $Bu \in L^2_{loc}(\mathbf{R}^n)$. Henceforth we may assume that $X = \mathbf{R}^n$ and $x_0 = 0$.

From Ben's talk (Thm 3.1.3 Hormander 1), we may make a change of coordinates so that

(1) $\Lambda = \{x = H'(\xi)\}$ where H is homogeneous of degree 1, and
(2) $\chi(x, \xi) = (x - H'(\xi), \xi)$.

Then

$$\chi\Gamma \subseteq N^*0 = 0 \times (\mathbf{R}^n \setminus 0).$$

This is the conormal bundle of 0 and therefore by my talk (Hormander 1, Prop 2.4.1) we maya assume that u is a Fourier integral operator with linear phase. That is, there is a symbol $a \in S^{m-n/4}(\mathbf{R}^n \setminus 0)$ such that

$$u(x) = \int_{\mathbf{R}^n \setminus 0} e^{-i\langle x, \theta\rangle} a(\theta)\ d\theta.$$

That is, u is the Fourier transform of a (of course this makes sense even if $a \notin L^1$). Since $WF(u)$ is a small neighborhood, u is rapidly decaying at infinity, so $u \in L^2_{loc}$ iff $u \in L^2$. Multiplying by $e^{-i\langle x,\theta\rangle}$ is a Fourier integral operator of order 0 so by Parseval's formula $u \in L^2$ iff $m - n/4 < -n/2$ (given that u is elliptic at $(0, \xi_0)$, otherwise we can pass to a weaker symbol class), thus $m < -n/4$, as desired. □

Theorem 5.2 (characterization of Hilbert-Schmidt operators)**.** *A Fourier integral operator $A : \mathcal{D}'(Y) \to \mathcal{D}'(X)$ is Hilbert-Schmidt iff $A \in I^m_\rho(X \times Y)$ with*

$$m < -\frac{\dim X + \dim Y}{4}$$

and the Schwartz kernel of A has compact support.

Proof. The proper support in $T^*(X \times Y)$ corresponds to compact support of the Schwartz kernel in $X \times Y$. Moreover $A \in I^m_\rho(X \times Y)$ iff $A \in H_{-m-n/4}$ where $n = \dim X + \dim Y$, and $A \in H_0$ iff $A \in L^2$ iff A is Hilbert-Schmidt (since the Hilbert-Schmidt norm is the L^2 norm, and A has compact support). □